Math 142 Lecture Notes
Algebraic Topology

Professor: Jamie Conway
Scribe: Daniel Raban

Contents

1 Introduction to Topology

1.1 Motivation and overview

What is topology? It is not geometry. Geometry is the study of "rigid" objects, distances, curvature, and symmetries/isometries. On the other hand, topology is the study of "non-rigid" phenomena, connectivity, and deformation (stretching, squeezing, etc.).

Example 1.1. Consider the difference between the surface of a tennis ball and the surface of a soccer ball. Geometrically, the surfaces have different properties, but a topological point of view would call them similar, or "the same."

Example 1.2. Consider the difference between A, $\mathcal{A}$, and $\mathscr{A}$. Geometrically, these are different shapes, but you might think of them all as the letter A. There is some common property they all share that makes them appear like the shape of the letter A.

Example 1.3. Are the letters V and X the the same, topologically? Maybe not. You can remove a point from the X and get 4 pieces, but you cannot do that with the V, no matter how you stretch it.

What does "the same" mean? We will see two approaches to this:

1. "homeomorphism" (think reparameterising)
2. "homotopy equivalent" (think same number of holes).

Example 1.4. A metal washer and a toilet paper roll might be considered to have the same number of holes.

Where does algebra come in? The idea is to encode information about your space using algebra.

Example 1.5. We will find a map G that takes a topological space and associates a group. Ideally, we want this to "respect" maps between spaces. So if $f : X \to Y$ is a continuous map, then $f_* : G(X) \to G(Y)$ will be a homomorphism. We also want composition to carry through; i.e. if $f : X \to Y$ and $g : Y \to Z$ are continuous, then $(g \circ f)_* = g_* \circ f_*$.

In this course, we will see two algebraic invariants:

1. Fundamental group $\pi_1(X)$
2. Homology groups $H_*(X)$.

They will fit in to the following outline of the course (probably):

1. Point-set topology (up to midterm 1)

2. Fundamental group
3. Homology
4. Applications.

There are other approaches to topology. For example, people study differential topology and analysis on topological spaces, which are both rich fields in their own right.

1.2 Topological spaces

Definition 1.1. Let X be a set. Then a *topology* on X is a collection of subsets of X, called *open sets*, such that

1. The sets $\varnothing$ and X are both open.
2. Any union of open sets is an open set (even of uncountably infinitely many).
3. The intersection of finitely many open sets is open.

The set X, along with its topology is called a *topological space*.

Example 1.6. This is the usual topology on $\mathbb{R}^n$ (also called $\mathbb{E}^n$).[1] Call a set $A \subseteq \mathbb{R}^n$ open iff for all $x \in A$, we can find an $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq A$, where $B_\varepsilon(x)$ is the ball of radius ε centered at x.

Let's check the properties.

1. $\varnothing$ is vacuously open, and $\mathbb{R}^n$ itself is open because for $x \in \mathbb{R}^n$, $B_\varepsilon(x) \subseteq \mathbb{R}^n$ for all ε.
2. If $A = \bigcup A_i$ and $x \in A$, then $x \in A_i$ for some i. A_i is open so there exists some $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq A_i$. So $B_\varepsilon(x) \subseteq A_i \subseteq \bigcup A_i = A$.
3. If $A = \bigcap_{i=1}^n A_i$ and $x \in A$, then $a \in A_i$ for $i = 1, \dots, n$. So for each i, there exists ε_i such that $B_{\varepsilon_i}(x) \subseteq A_i$. Let $\varepsilon = \min(\varepsilon_1, \dots, \varepsilon_n)$. Then $B_\varepsilon(x) \subseteq B_{\varepsilon_i}(x) \subseteq A_i$ for all i. So $B_\varepsilon(x) \subseteq \bigcap_{i=1}^n A_i = A$, making A open.

Example 1.7. A metric space (X, d) automatically has a topology *induced* by the metric. For $x \in X$, define $B_\varepsilon(x) = \{y \in X : d(x, y) \leq \varepsilon\}$.[2] Then define the topology on X as in the $\mathbb{R}^n$ example.

Remark 1.1. Different metrics might give the same topology.

Topologies induced by metrics are easier to visualize. However, there are "weirder" topologies that do not necessarily correspond to a metric.

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[1]The letter E here is for Euclidean.

[2]We could have used $<$ here instead of $\leq$, but it does not matter because they produce the same topology.

Example 1.8. Let X be an space, and let the open sets be $\{\varnothing, X\}$. This is called the *trivial* or *indiscrete topology.*

Example 1.9. Let X be any space, say that every subset of X is open. This is called the *discrete topology.*

Example 1.10. If X is a topological space and $Y \subseteq X$, then the *subspace* (or *induced*) *topology* on Y has $A \subseteq Y$ open iff $A = Y \cap U$ for some $U \subseteq X$ open.

Definition 1.2. Let X be a topological space. A set $B \subseteq X$ is *closed* if $X \setminus B$ is open.[3]

Example 1.11. Both $\varnothing$ and X are closed, in addition to being open.

Definition 1.3. If $x \in X$, a *neighborhood* of x is any open set $A \subseteq X$ with $x \in A$.[4]

How do we show a set A is closed? Show that $X \setminus A$ is open. How do we show that a set A is open? For every $x \in A$, find a neighborhood U_x such that $U_x \subseteq A$. Then $A = \bigcup_{x \in A} U_x$ is a union of open sets, so it is open.

[3]This is sometimes called $X - B$, and is $\{x \in X : x \notin B\}$.

[4]This term adds nothing new, but it is shorter and cleaner to say and write. Otherwise, we would always have to talk about "an open set containing x."

2 Limit Points, Closure, and Continuity

2.1 Limit points and closure

Definition 2.1. Let $A \subseteq X$ and $p \in X$. Then p is a *limit point* of A if every neighborhood U of p satisfies $U \cap (A \setminus \{p\}) \neq \varnothing$; i.e. U has a point of A besides p.

Remark 2.1. A limit point of a set may not be contained in the set.

Theorem 2.1. *$A \subseteq X$ is closed iff A contains all its limit points.*

Proof. ($\Longrightarrow$) If X is closed, then $X \setminus A$ is open. So for any $x \in X \setminus A$, $X \setminus A$ us a neighborhood of x. But $(X \setminus A) \cap (A \setminus \{x\}) = \varnothing$. So x is not a limit point of A. So any limit point of A is in A.

($\Longleftarrow$) If A contains all its limit points, we want to show that $X \setminus A$ is open. If $x \in X \setminus A$, it is not a limit point, so there exists a neighborhood U_x of x with $U_x \cap (A \setminus \{x\}) = \varnothing$. So $U_x \subseteq X \setminus A$. Then $X \setminus A = \bigcup_{x \in X \setminus A} U_x$ is a union of open sets making it open. So A is closed. □

Definition 2.2. If $A \subseteq X$, the *closure* of A is

$$\overline{A} := A \cup \{\text{limit points of A}\}.$$

Theorem 2.2. *$\overline{A}$ is the smallest closed set containing A.*

Proof. If $A \subseteq B \subseteq X$ and B is closed, any limit point of A is a limit point of B. B is closed, so B contains all its limit points; then B contains all the limit points of A. So $\overline{A} \subseteq B$.

We need to show that $\overline{A}$ is closed. Let $x \in X \setminus \overline{A}$; then x is not a limit point of A. So there exists a neighborhood U_x of x such that $U_x \subseteq X \setminus A$. We want to show that $U_x \subseteq X \setminus \bar{A}$. If $y \in U_x$ is a limit point of A, then U_x is a neighborhood of y, and $U_x \cap A = \varnothing$. But y is a limit point, so such a neighborhood shouldn't exist. So $U_x \cap \overline{A} = \varnothing$; i.e. $U_x \subseteq X \setminus A$. So $X \setminus \overline{A} = \bigcup_{x \in X \setminus \overline{A}} U_x$, making it open. So $\bar{A}$ is closed. □

Corollary 2.1. *$A \subseteq X$ is closed iff $A = \overline{A}$.*

Definition 2.3. A *base* of a topological space X is a collection of open sets such that if $A \subseteq X$ is open, A is a union of open sets in the collection.

Example 2.1. $\mathbb{R}^n$ with the usual topology has base $\{B_\varepsilon(x) : x \in \mathbb{R}^n, \varepsilon > 0\}$.[5]

[5] Last lecture, we used this notation to mean closed balls. Here, we mean open, so we are using the "$<$" notation, rather than "$\leq$."

2.2 Continuity

Definition 2.4. A function $f : X \to Y$ is *continuous* if $f^{-1}(A) \subseteq X$ is open whenever $A \subseteq Y$ is open.[6]

A continuous function is often called a *map*.

Theorem 2.3. *If $A \subseteq X$ has the subspace topology, then the inclusion $i : A \to X$, sending $a \mapsto a$, is continuous.*

Proof. If $U \subseteq X$ is open, then

$$i^{-1}(U) = \{a \in A : i(a) \in U\} = A \cap U,$$

which is open by the definition of the subspace topology. □

Theorem 2.4. *If $f : X \to Y$ and $g : Y \to Z$ are continuous, then so is $g \circ f : X \to Z$.*

Proof. Note that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. If $A \subseteq Z$ is open, then $g^{-1}(A)$ is open, as g is continuous. So $f^{-1}(g^{-1}(A))$ is open, as f is continuous. This says that $(g \circ f)^{-1}(A)$ is open, so $g \circ f$ is continuous. □

Corollary 2.2. *If $f : X \to Y$ is continuous and $A \subseteq X$ has the subspace topology, then $f|_A : A \to Y$ is continuous, where $f|_A(a) = f(i(a))$.*

Theorem 2.5. *The following are equivalent.*

1. *$f : X \to Y$ is continuous.*
2. *$f^{-1}(A)$ is closed whenever $A \subseteq Y$ is closed.*
3. *If $\{U_\alpha\}$ is a base for the topology on Y, then $f^{-1}(U_\alpha)$ is open for all α.*

Proof. See textbook. □

Example 2.2. Let X be a set with the discrete topology, let Y be any set with any topology, and let $f : X \to Y$ be any function. Then f is continuous, as $f^{-1}(A) \subseteq X$ is always open for any subset $A \subseteq Y$.

Example 2.3. Continuity from anaylsis is the same as continuity in topology, when they both apply. If $f : (X, d_x) \to (Y, d_y)$ is a function of metric spaces, then f is "analysis continuous" if for all $x \in X$ and $\varepsilon > 0$, $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$. So if $A \subseteq Y$ is open, we want to show that $f^{-1}(A)$ is open. So if $x \in f^{-1}(A)$, then $f(x) \in A$. So if A open, there exists some $\varepsilon > 0$ such that $B_\varepsilon(f(x)) \subseteq A$. Since f is "analysis continuous," there exists a $\delta > 0$ such that $f(B_\delta(x)) \subseteq B_\varepsilon(f(x)) \subseteq A$. So $B_\delta(x) \subseteq f^{-1}(A)$, and then $f^{-1}(A)$ is open. So if f is "analysis continuous," f is "topology continuous." The converse is left as an exercise.

[6] While f might not have an inverse, we mean here that $f^{-1}(A) = \{x \in X : f(x) \in A\}$.

3 Homeomorphisms, Disjoint Unions, and Product Spaces

3.1 Homeomorphisms

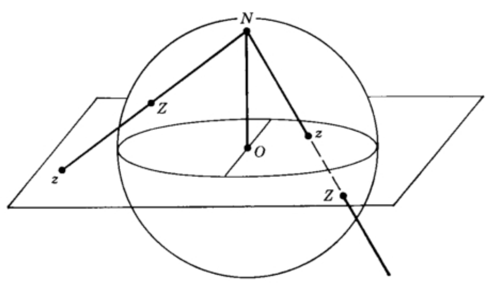
How can we say that two topological spaces are "the same"?

Definition 3.1. A function $f : X \to Y$ is a *homeomorphism* if f is a continuous bijection with a continuous inverse. We call X, Y *homeomorphic spaces*, denoted by $X \cong Y$.

If f is a homeomorphism with inverse f^{-1}, then if $A \subseteq X$ is open, then $(f^{-1})^{-1}(A) \subseteq Y$ is open (as f^{-1} is continuous). Since $f = (f^{-1})^{-1}$, this means that a homeomorphism is a bijection between the open sets in X and the open sets in Y.

Example 3.1. A continuous bijection might not have a continuous inverse. Let $X = \mathbb{R}$ with the discrete topology and $Y = \mathbb{R}$ with the trivial topology. Let $f : X \to Y$ be defined as $f(x) = x$. f is continuous, as $f^{-1}(\varnothing) = \varnothing$ is open, and $f^{-1}(\mathbb{R}) = \mathbb{R}$ is open. But $f^{-1} : Y \to X$ takes $f^{-1}(x) = x$, and $(f^{-1})^{-1}(\{1\}) = \{1\}$ is not open in Y.

Example 3.2 (stereographic projection)**.** Define the set $S^{(n)} = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}$ be the n-dimensional sphere.[7] Consider $f : S^n \setminus \{(0, 0, \dots, 0, 1)\} \to \mathbb{R}^n$ (the domain missing the "north pole") given as follows. Take x on the sphere and draw a line containing x and the north pole; this line intersects the plane, and we set $f(x)$ to be this point of intersection.[8] Check that this is a bijection.



We want to show that f is continuous. Let $U \subseteq \mathbb{R}^n$ be open. Let $U' \subseteq \mathbb{R}^{n+1}$ be all the half-lines from p to a point $x \in U$ (not including p). Check that U' is open in $\mathbb{R}^{n+1}$. S_n has the subspace topology, so $U' \cap S^n$ is open in S^n. But $f^{-1}(U) = U' \cap S^n$. So $f^{-1}(U)$ is open, making f continuous. A similar argument using U' shows that f^{-1} is continuous. So f is a homeomorphism.

3.2 Creating new topological spaces

Using the idea of the subspace topology, we can create new topological spaces form larger ones. How else can we construct topological spaces?

[7]The n-dimensional sphere sits in $n + 1$ dimensional space.
[8]I did not create this picture; I found it on Google.

3.2.1 Disjoint unions of spaces

Definition 3.2. If X, Y are topological spaces, then the *disjoint union* $X \amalg Y$ (also called $X + Y$) is the set $X \amalg Y$ with open sets U_α and V_β, where $U_\alpha \subseteq X$ is open, $V_\beta \subseteq Y$ is open, and unions of these sets are open.

Example 3.3. Let $X = \{1, 2, 3\}$ with open sets $\varnothing, X$, and let $Y = \{3, 4, 5\}$ with open sets $\varnothing, Y, \{3, 4\}$. Then

$$X \amalg Y = \{1, 2, 3_x, 3_y, 4, 5\},$$

with open sets $\varnothing, \{1, 2, 3_x\}, \{3_y, 4, 5\}, \{3_y, 4\}, \{1, 2, 3_x, 3_y, 4\}, \{1, 2, 3_x, 3_y, 4, 5\}$.

3.2.2 Products of spaces

Definition 3.3. If X and Y are topological spaces, then the *product space* $X \times Y$ is the set

$$X \times Y = \{(x, y) \in X \times Y : x \in X, y \in Y\}$$

with a base for the topology given by $\{U \times V : U \subseteq X \text{ open}, V \subseteq Y \text{ open}\}$.

Example 3.4. $\mathbb{R}^2 \cong \mathbb{R} \times \mathbb{R}$. Here, the set $(0, 1) \times (0, 1)$ is open and in the base. The open unit ball is an open set, but it is not in the base; it is a union of infinitely many squares in the base.

Product spaces come with projection maps $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$, where $p_1(x, y) = x$, and $p_2(x, y) = y$.

Theorem 3.1. *If $X \times Y$ has the product topology, then p_1 and p_2 are continuous, and take open sets to open sets. Furthermore, the product topology is the smallest topology for which p_1 and p_2 are continuous.*

Proof. If $U \subseteq X$ is open, then $p_1^{-1}(U) = U \times Y$. But U is open in X and Y is open in Y, so $U \times Y$ is open in $X \times Y$. So p_1 is continuous. Similarly, p_2 is continuous.

If $A \subseteq X \times Y$ is open, then $A = \bigcup (U_i \times V_i)$ for some open sets $U_i \subseteq X$ and $V_i \subseteq Y$. Then

$$p_1(A) = \bigcup p_1(U_i \times V_i) = \bigcup U_i,$$

which is a union of open sets, making it open in X. The same argument works for p_2.

Now assume $X \times Y$ has another topology where p_1, p_2 are continuous. Then if $U \subseteq X$ and $V \subseteq Y$ are open, then $p_1^{-1}(U) = U \times Y$ and $p_2^{-1}(V) = X \times V$ are open in this topology. So $(U \times Y) \cap (X \times V) = U \times V$ is open, and then any union $\bigcup (U_i \times V_i)$ is open in this topology. So any open set in the product topology is open in this new topology. $\square$

Theorem 3.2. *A function $f : Z \to X \times Y$ is continuous iff $p_1 \circ f$ and $p_2 \circ f$ are continuous.*

Proof. ($\implies$) If f is continuous, then $p_1 \circ f$ and $p_2 \circ f$ are compositions of continuous functions and are therefore continuous.

($\impliedby$) If $p_1 \circ f$ and $p_2 \circ f$ are continuous, we need to show that $f^{-1}(U \times V) \subseteq Z$ is open for any open $U \subseteq X$, $V \subseteq Y$. But

$$\begin{aligned} f^{-1}(U \times V) &= f^{-1}(p_1^{-1}(U) \cap p_2^{-1}(V)) \\ &= f^{-1}(p_1^{-1}(U)) \cap f^{-1}(p_2^{-1}(V)) \\ &= (p_1 \circ f)^{-1}(U) \cap (p_2 \circ f)^{-1}(V), \end{aligned}$$

which is an intersection of open sets since $p_1 \circ f$ and $p_2 \circ f$ are continuous. So it is open. $\square$

4 Identification Spaces and Attaching Maps

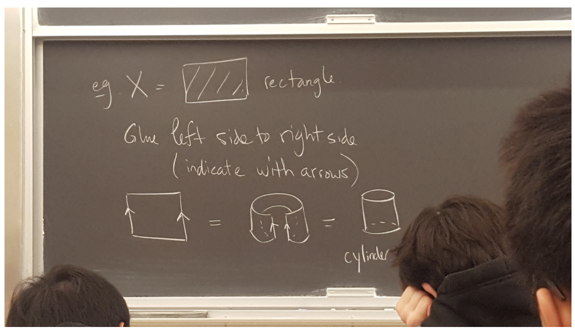
4.1 Identification spaces

How do we construct new topological spaces? We have already covered

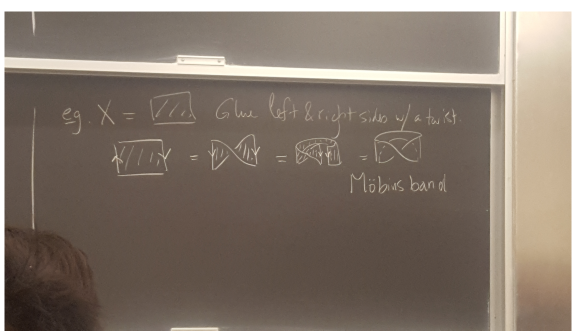
1. subspaces
2. disjoint unions
3. product spaces[9].

We will add identification spaces to the list. The ideas is that we start with a topological space X and "identify"/"set equal"/"glue" some subsets.

Example 4.1. Let X be a rectangle, and glue the left side to the right side. We indicate the gluing with arrows. Here, we get a cylinder.

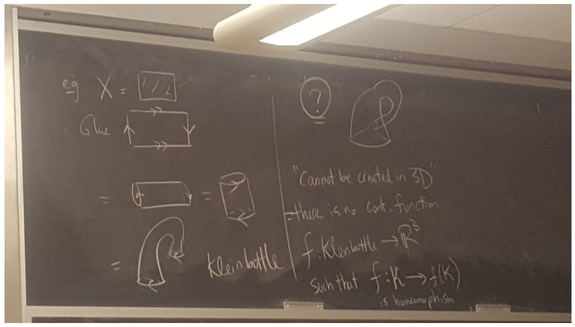


Example 4.2. Let X be a rectangle, and glue the left and right sides, but with a twist. We indicate this with the arrows on our diagram. We get a Möbius band.

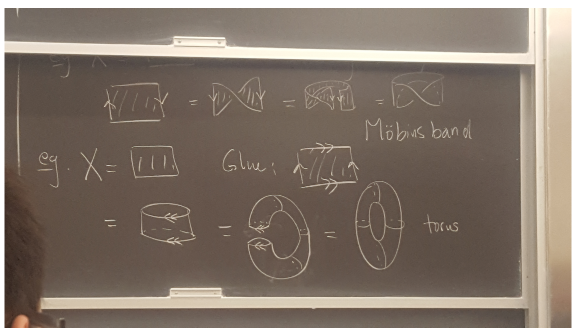


[9]What we called the "product topology" is actually the *box topology*, but these two coincide for products of finitely many spaces.

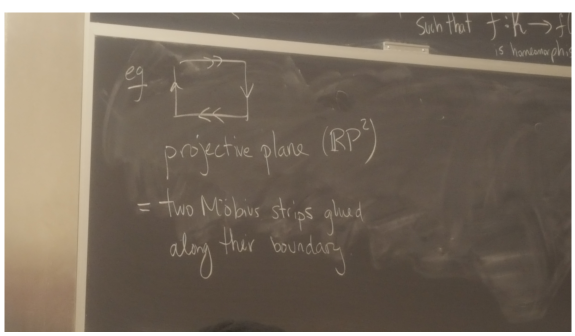
Example 4.3. Let X be a rectangle, and glue the left and right sides with no twists. The, glue the top and bottom together with no twists. We get a torus.



Example 4.4. Let X be a rectangle, and glue the top and bottom the same way, but glue the left and right sides together with a twist. We get a Klein bottle, but this "cannot be created in 3D." More precisely, there is no continuous function f : Klein bottle $\to \mathbb{R}^3$ such that $f : K \to f(K)$ is a homeomorphism.



Example 4.5. Let X be a rectangle, and glue the top and bottom with a twist and the left and right sides together with a twist. We get something called the "projective plane ($\mathbb{R}P^2$), which is two Möbius strips glued along their boundary. This also cannot be created in 3D.



Let's give a more formal definition.

Definition 4.1. If X is a topological space, let a partition $\mathcal{P}$ be a collection of nonempty subsets of X such that each $x \in X$ is in exactly one subset $A_x \in \mathcal{P}$. Write $\pi : X \to P$ sending $x \mapsto A_x$. Then make a new space Y (the *identification space*), by setting the points of Y to be elements in P, and $A \subseteq Y$ is open iff $\pi^{-1}(A) \subseteq X$; i.e. π is actually a map $\pi : X \to Y$, and the topology on Y is the largest so that π is continuous. This is the *identification topology.*

Example 4.6. Look at the uniq square $[0,1] \times [0,1] \subseteq \mathbb{R}^2$. To make a cylinder, set P to include the subsets:

- one singleton subset $\{x\}$ for each $x \in (0,1) \times [0,1]$
- $\{(0,y),(1,y)\}$ for each $y \in [0,1]$

Remark 4.1. In some of our other examples, we need to also put all four corners of the rectangle into one subset.

Theorem 4.1. *If Y is an identification space, and Z is any space, then $f : Y \to Z$ is continuous iff $f \circ \pi : X \to Z$ is continuous.*

4.2 Attaching maps

Definition 4.2. Let X, Y be topological spaces, $A \subseteq X$ be a subspace, and $f : X \to Y$ be a continuous map. Start with $X \amalg Y$, and let $\mathcal{P}$ have the subsets

- $f^{-1}(y) \cup \{y\}$ for $y \in f(A)$
- $\{x\}$ for each $x \in X \setminus A$
- $\{y\}$ for each $y \in T \setminus f(A)$.

We call the identification space $X \cup_f Y$; here f is called the *attaching map.*

Here is a special example of this construction.

Definition 4.3. Let X be any space, $A \subseteq X$, $Y = \{*\}$ (a space containing only 1 point), and $f : A \to Y$ be $a \mapsto *$.So $\mathcal{P}$ has

- $\{x\}$ for $x \in X \setminus A$
- $A \cup \{*\}$.

The identification space $X \cup_f Y$ is called the *quotient space* X/A.

Here, we have crushed A to a point.

Example 4.7. Let X be an interval and A be the boundary (the two endpoints). Then X/A is the circle S^1.

Example 4.8. Let X be a disc and A be the boundary (a circle). Then X/A is the sphere S^2.

You might have more trouble believing this. Think of bending your disc into the shape of the sphere, missing a patch at the top. If we condense the boundary to a single point, this closes the sphere.

Example 4.9. Let $X = B^n$ be an n-dimensional ball and $A = S^{n-1}$ be its boundary. Then $X/A \cong S^n$.

Remark 4.2. While these pictures may help with intuition, they are not exactly precise. We are not actually bending anything in our construction; we are identifying points together.

Theorem 4.2. *If $f : X \to Y$ is continuous and surjective, and if f maps open sets to open sets (or closed sets to closed sets), then Y is an identification space, and f is the projection map (π).*

Proof. Define a partition $\mathcal{P}$ of X that has subsets $f^{-1}(y)$ for each $y \in Y$. Here, surjectivity implies that $f^{-1}(y) \neq \varnothing$ for every y. We want to show that the identification space from $\mathcal{P}$ is homeomorphic to Y; i.e. we want to show that the topology on Y is the larges so that f is continuous. In other words, we need to show that if $f^{-1}(A) \subseteq X$ is open, then $A \subseteq Y$ is open.

Suppose f takes open sets to open sets. Since f is surjective, $f(f^{-1}(A)) = A$. So if $f^{-1}(A)$ is open, then $A = f(f^{-1}(A))$ is open by hypothesis. The case of f sending closed sets to closed sets is similar, except it includes taking complements. $\square$

Next time, we will show that if

$$X = B^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 \leq 1\},$$

$$A = S^{n-1} := \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = 1\},$$

then $B^n/S^{n-1} \cong S^n$.

Here's something to think about before next lecture: crushing S^{n-1} to a point is the same as gluing 2 copies of B^n together along their boundaries. Why?

5 Quotient Spaces and Introduction to Compactness

5.1 Quotient spaces (continued)

If we have a $f : X \to Y$ is surjective, then we can define a partition $\mathcal{P}$ on X by taking sets $f^{-1}(y)$ for each $y \in Y$. So we can define an identification space Y^* from X and $\mathcal{P}$. Last time, we proved the following theorem.

Theorem 5.1. *If $f : X \to Y$ is continuous, surjective, and maps open sets to open sets, then $Y^* \cong Y$.*

We said last time that a quotient space X/A was the special case where $A \subseteq X$, $Y = \{*\}$, $f : A \to Y$, and X/A is the identification space $X \cup_f Y$.

Proposition 5.1. *For any $n \in \mathbb{N}$ with $n > 1$, $B^n/S^{n-1} \cong S^n$.*

Proof. Recall from stereographic projection that, calling p the "north pole" on S^n, $S^n \setminus \{p\} \cong \mathbb{R}^n$. Let $g : \mathbb{R}^n \to S^n \setminus \{p\}$ be a homeomorphism. We also have that $B^n \setminus S^{n-1} \cong \mathbb{R}^n$ by the homeomorphism $h : B^n \setminus S^{n-1} \to \mathbb{R}$ given by $x \mapsto (1 + \tan(\|x\|\, \pi/2))x$. Show that this is a homeomorphism (exercise).

Then define $f : B^n \to S^n$ by

$$f(x) = \begin{cases} p & x \in S^{n-1} = \partial B^n \text{(boundary of } B^n) \\ g(h(x)) & x \notin S^{n-1} \quad \text{(i.e. } x \in (\mathring{B}^n)). \end{cases}$$

Show that f is continuous (exercise). Show that f takes open sets to open sets (also exercise, but similar to the previous). Then the previous theorem implies that Y^* from f is homeomorphic to S^n. But

$$f^{-1}(y) = \begin{cases} \text{singleton} & y \neq p \\ S^{n-1} & y = p, \end{cases},$$

so $Y^* = B^n/S^{n-1}$. □

5.2 Compactness

5.2.1 Open covers and compactness

Definition 5.1. An *open cover* of a topological space X is a collection[10] of open sets $\{A_i\}$ with $A_i \subseteq X$ such that $X = \bigcup_i A_i$. If $\{A_i\}$ and $\{B_j\}$ are open covers of X, and $\{B_j\} \subseteq \{A_i\}$, then $\{B_j\}$ is called a *subcover* of $\{A_i\}$.

[10]This collection need not even be countable. We may have an uncountable collection of open sets in our cover.

Example 5.1. Let X be any space. Then $\{X\}$ is an open cover.

Example 5.2. Let $X = \mathbb{R}$, and take the collection $\{A_i\} := \{B_\varepsilon(x) : \varepsilon > 0, x \in \mathbb{R}\}$; this is an open cover. Let $\{B_j\} := \{B_\varepsilon(x) : \varepsilon > 0, \varepsilon \in \mathbb{Q}, x \in \mathbb{Q}\}$; then $\{B_j\}$ is a subcover of $\{A_i\}$.

Definition 5.2. A space X is *compact* if every open cover of X has a finite subcover.[11]

Example 5.3. We show that $\mathbb{R}$ with the usual topology is not compact. Let $X = \mathbb{R}$ and $A_i = (i-1, i+1)$ for each $\mathbb{Z}$. $\{A_i\}$ is an open cover of $\mathbb{R}$, but for $i \in \mathbb{Z}$, $i \in A_j \implies i = j$. So there are no subcovers of $\{A_i\}$; in particular, there are no finite subcovers. Similarly, $\mathbb{R}^n$ is not compact.

Definition 5.3. A subset $A \subseteq X$ is *compact* if it is compact with the subspace topology.

Theorem 5.2. *If $f : X \to Y$ is continuous, and X is compact, then the image $f(X)$ is compact.*

Proof. Assume that f is surjective; if not, just consider $g : X \to f(X)$ given by $g(x) = f(x)$. Let $\{A_i\}$ be an open cover of $Y = f(X)$. Since f is continuous, $f^{-1}(A_i)$ is open for each A_i, and $\forall x \in X$, $x \in f^{-1}(A_i)$ for some i (as $\{A_i\}$ is a cover for Y). So $\{f^{-1}(A_i)\}$ is an open cover of X, and by the compactness of X, there exists a finite subcover of X; i.e. $X = f^{-1}(A_{i_1}) \cup \cdots \cup f^{-1}(A_{i_n})$. Since $f(f^{-1}(A_i)) = A_i$, we have $Y = f(X) = A_{i_1} \cup \cdots \cup A_{i_n}$. So $\{A_{i_1}, \ldots, A_{i_n}\}$ is a finite subcover of $\{A_i\}$. Since $\{A_i\}$ was an arbitrary open cover, this works for every cover. Hence, Y is compact. $\square$

Here is the flow of the previous proof in a picture:

$$\begin{tikzcd}
X & Y \\
\text{open cover} \arrow[d] & \text{open cover} \arrow[l] \\
\text{finite subcover} \arrow[r] & \text{finite subcover.}
\end{tikzcd}$$

The following theorem has a similar structure to its proof.

Theorem 5.3. *If X is compact, and $B \subseteq X$ is closed, then B is compact.*

Proof. Let $\{A_i\}$ be an open cover of B. Then each $A_i = A'_i \cap B$ for some $A'_i \subseteq X$ open, and $B \subseteq \bigcup A'_i$. Note that $\{A'_i\} \cup \{X \setminus B\}$ is an open cover of X; $X \setminus B$ is open because B is closed. X is compact, so there exists a finite subcover $X = A'_{i_1} \cup \cdots A'_{i_n} \cup (X \setminus B)$; the set $X \setminus B$ may not be necessary, but it has empty intersection with B, so it doesn't matter if we keep it. Then $B = A_{i_1} \cup \cdots \cup A_{i_n}$, and since $\{A_i\}$ was a generic open cover, we conclude that B is compact. $\square$

[11] Compactness is a property of the space itself, not of a particular cover.

5.2.2 Hausdorff Spaces

Definition 5.4. A space X is *Hausdorff* if for all $x, y \in X$ with $x \neq y$, there are neighborhoods U_x of x and U_y of y such that $U_x \cap U_y = \varnothing$.

Theorem 5.4. *If X is Hausdorff, and $A \subseteq X$ is compact, then A is closed.*

We will delay proof of this until next time. For now, we will use this theorem to prove the following theorem.

Theorem 5.5. *If $f : X \to Y$ is a continuous bijection, X is compact, and Y is Hausdorff, then f is a homeomorphism.*

Proof. If f takes closed sets to closed sets, then f^{-1} is continuous, and we are done. If $B \subseteq X$ is closed, then B is compact. The function f is continuous, so $f(B) \subseteq Y$ is compact. Then, by the previous theorem, $f(B)$ is closed. $\square$

6 Compactness and Analysis

6.1 Compact subsets of Hausdorff spaces

Here is the theorem we promised to prove last time.

Theorem 6.1. *If X is Hausdorff, and $A \subseteq X$ is compact, then A is closed.*

Proof. We will show that $X \setminus A$ is open. Let $x \in X \setminus A$, and choose $z \in A$. X s Hausdorff, so there exist neighborhoods U_z, V_z such that $x \in U_z$, $z \in V_z$, and $U_z \cap V_z = \varnothing$. We can vary z to get a collection $\{V_z : z \in A\}$ of open sets. Then $\{V_z \cap A\}$ is an open cover of A. A is compact, so $A = (A \cap V_{z_1}) \cup \cdots \cup (A \cap V_{z_n})$ for some $z_1, \ldots, z_n \in A$. This implies that $A \subseteq V_{z_1} \cup \cdots \cup V_{z_n} = V$.

Since $U_{z_i} \cap V_{z_i} = \varnothing$, we know that $U = U_{z_1} \cap \cdots \cap U_{z_n}$ is disjoint from V. So $U \cap A = \varnothing$; i.e. $U \subseteq X \setminus A$. Also, U is open (as an intersection of finitely many open sets), and $x \in U$, so we have an open neighborhood of x that is contained in $X \setminus A$. Since x was any point in $X \setminus A$, we conclude that $X \setminus A$ is open. Hence, A is closed. □

6.2 Generalizations of theorems from analysis

6.2.1 The Bolzano-Weiertrass theorem

Recall the following theorem from analysis.

Theorem 6.2 (Bolzano-Weierstrass)**.** *Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.*

Given a sequence (a_n), we can construct the set $\{a_n : n \in \mathbb{N}\} \subseteq \mathbb{R}$. We can think of the limit of a sequence as a limit point of $\{a_n\}$ (if $\{a_n\}$ infinite). This gives rise to a more general topological analogue of the Bolzano-Weierstrass theorem.

Theorem 6.3 (Bolzano-Weierstrass)**.** *If X is compact, and $A \subseteq X$ is infinite, then A has a limit point.*

Proof. In pursuit of a contradiction, assume A has no limit points; we will show that A must be finite. Given $x \in X$, it is not a limit point of A. So if $x \in A$, then we can find a neighborhood U_x of x such that $U_x \cap (A \setminus \{x\}) = \varnothing$; i.e. $U_x \cap A = \{x\}$. Likewise, if $x \notin A$, we can find a neighborhood U_x of x such that $U_x \cap (A \setminus \{x\}) = \varnothing$; i.e. $U_x \cap A = \varnothing$. Then $\{U_x\}$ is an open cover of X, so $X = U_{x_1} \cup \cdots \cup U_{x_n}$, as X is compact. But $A = A \cap X = (A \cap U_{x_1}) \cup \cdots (A \cap U_{x_n})$. We had that $|A \cap U_{x_i}| \leq 1$ for each i, so $|A| \leq n < \infty$. This is a contradiction, as A was assumed to be infinite. □

6.2.2 Characterization of compactness

Theorem 6.4. *If $A \subseteq \mathbb{R}^n$ is compact, then it is closed and bounded.*

Proof. $\mathbb{R}^n$ is Hausdorff, so since A is a compact subset, it is closed. Since $A \subseteq \bigcup_{n=1}^{\infty} B_n(0)$, $\{A \cap B_n(0)\}$ is an open cover of A. A is compact, so $A = (A \cap B_{n_1}(0)) \cup \cdots \cup (A \cap B_{n_k}(0))$. Take $N = \max\{n_1, \dots, n_k\}$. Then $A = A \cap B_N(0)$; i.e. $A \subseteq B_N(0)$, so it is bounded. $\square$

Recall the following theorem from analysis.

Theorem 6.5. *If $f : [a, b] \to \mathbb{R}$ is continuous, then f is bounded and attains its bounds.*

In the general topological setting, this becomes the following theorem.

Theorem 6.6. *If X is compact, and $f : X \to \mathbb{R}$ is continuous, then f is bounded and attains its bounds.*

Proof. The image $f(X)$ is compact, so $f(X)$ is closed and bounded by the theorem we just proved. Since $f(X)$ is bounded and nonempty, it has a supremum S and an infimum I. We know that S and I are limit points of $f(X)$ (if $f(X)$ is finite, the supremum is just one of the points). The set $f(X)$ is closed, so it contains its limit points. So $S, I \in f(X)$; i.e. $S = f(x_0)$ and $I = f(x_1)$ for some $x_0, x_1 \in X$, so f attains its bounds. $\square$

6.3 Tychonoff's product theorem (finite version)

We want to prove the converse to the previous theorem that sats compact $\implies$ clsoed and bounded in $\mathbb{R}^n$. To do that, we will establish a more general theorem about compactness of product spaces. First, we need a lemma.

Lemma 6.1. *If $\{U_i\}$ is a base for the topology of a space X, then X is compact iff every open cover $\mathcal{C}$ of X such that $\mathcal{C} \subseteq \{U_i\}$ has a finite subcover.*

Proof. ($\implies$) This follows from the definition of compactness.

($\impliedby$) Let $\mathcal{C}$ be any open cover of X, and let $\mathcal{B}$ be a base. We build a new open cover $\mathbb{C}'$. For each $A \in \mathcal{C}$, $A = \bigcup_i U_i$, where $U_i \in \mathcal{B}$. Let $\mathcal{C}' := \{U_i \in \mathcal{B} : \exists A \in \mathcal{C} \text{ such that } U_i \subseteq A\}$. By assumption, $\mathcal{C}'$ has a finite subcover $\{U_{i_1}, \dots, U_{i_n}\}$. For each $i = 1, \dots, n$, $U_i \subseteq A_i$ for some $A_i \in \mathcal{C}$, so $X = \bigcup_{i=1}^n U_i \subseteq \bigcup_{i=1}^n A_i \subseteq X$. So $\bigcup_{i=1}^n A_i = X$ and $\mathcal{C}$ has a finite subcover. $\square$

Theorem 6.7 (Tychonoff (finite version))**.** *$X \times Y$ is compact iff X and Y are compact.*

Proof. ($\implies$) We have continuous functions $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ that are surjective. So $X = p_1(X \times Y)$ and $Y = p_2(X \times Y)$ are compact.

($\impliedby$) Let $\mathcal{C} = \{U_i \times V_i\}$ be an open cover of $X \times Y$ by open sets in the base of the product topology from the definition. We will show that $\mathcal{C}$ has a finite subcover, and then we will use the lemma.

If $x \in X$, then $p_2|_{\{x\}\times Y} : \{x\} \times Y \to Y$ is a homeomorphism. Since Y is compact, so is $\{x\} \times Y$. So there exists a subcover $\mathcal{C}_x \subseteq \mathcal{C}$ such that $\mathcal{C}_x = \{U_1^x \times V_1^x, \ldots, U_{n_x}^x \times V_{n_x}^x\}$ is finite, and $\{x\} \times Y \subseteq \bigcup_{i=1}^{n_x} U_i^x \times V_i^x$.

If $U^x = U_1^x \cap \cdots \cap U_{n_x}^x$, then $U^x \times Y \subseteq \bigcup_{i=1}^{n_x} U_i^x \times V_i^x$. So for every $x \in X$, we get an open set $U^x \subseteq X$; this makes $\{U^x : x \in X\}$ an open cover of X. X is compact, so $X = U^{x_1} \cup \cdots \cup U^{x_s}$. Then

$$X \times Y = \bigcup_{j=1}^{s} U^{x_j} \times Y = \bigcup_{j=1}^{s} \bigcup_{i=1}^{n_{x_j}} U_i^{x_j} \times V_i^{x_j}.$$

This is a finite union, so $\mathcal{C}$ has a finite subcover. $\square$

7 Compactness and Connectivity in $\mathbb{R}^n$

7.1 The Heine-Borel theorem and compactness in $\mathbb{R}^n$

Theorem 7.1 (Heine-Borel)**.** *Any closed and bounded interval $[a, b] \subseteq \mathbb{R}$ is compact.*

Proof. Give $[a, b] \subseteq \mathbb{R}$ the subspace topology, and let $\mathcal{C}$ be an open cover of $[a, b]$. Let $X = \{x \in [a, b] : [a, x]$ is contained in the union of finitely many elements of $\mathcal{C}\}$. If $b \in X$, then $[a, b] = U_1 \cup \cdots \cup U_n$ for $U_i \in \mathcal{C}$, so $\{U_1, \dots, U_n\}$ is a finite subcover of $\mathcal{C}$.

Think of $X \subseteq \mathbb{R}$. We know that $a \in X$, and $[a, a] = \{a\}$ is contained in some $U \in \mathcal{C}$ such that $a \in U$. Additionally, X is bounded above by b. So X has a supremum $s \in \mathbb{R}$. We want to show that $s = b$ and that $s \in X$.

Certainly, $s \leq b$, so $s \in [a, b]$. Let $U \in \mathcal{C}$ be an open set such that $s \in U$. If $s < b$, then we can find some $\varepsilon > 0$ such that $(s - \varepsilon, s + \varepsilon) \subseteq U$. If $s = b$, then we can find some $\varepsilon > 0$ such that $(s - \varepsilon, s] \subseteq U$; this set is also open in the subspace topology on $[a, b]$. We can find points of X arbitrarily close to s; i.e. we can find $x_\varepsilon \in X$ such that $|s - x| < \varepsilon/2$. If $x_\varepsilon \in X$, then $[a, x_\varepsilon] \subseteq U_1 \cup \cdots U_n$ for some $U_i \in \mathcal{C}$. if $s < x_\varepsilon$, then $[a, s] \subseteq [a, x_\varepsilon]$, so $s \in X$. If $s > x_\varepsilon$, then $[x_\varepsilon, s] \subseteq U$. So $[a, s] \subseteq U_1 \cup \cdots \cup U_n \cup U$, which makes $s \in X$.

Also, if $s < b$, then $[a, s + \varepsilon/2] \subseteq U_1 \cup \cdots \cup U_n \cup U$. So $s + \varepsilon/2 \in X$, contradicting the fact that s is the supremum of X. So $s = b$, which shows that $\mathcal{C}$ has a finite subcover. Since $\mathcal{C}$ was arbitrary, we conclude that $[a, b]$ is compact. $\square$

This implies the following theorem, which is more our end-goal.

Theorem 7.2. *$A \subseteq \mathbb{R}^n$ is compact iff A is closed and bounded.*

Proof. ($\Longrightarrow$) We proved this last lecture.

($\Longleftarrow$) A is bounded, so $A \subseteq [-s, s]^n$ for some $s > 0$. Let $C = [-s, s]^n$. The set $[-s, s]$ is compact in $\mathbb{R}$ by our previous theorem, so our product theorem for compact spaces says that $C \subseteq \mathbb{R}^n$ is compact. Then $A \subseteq C$ is closed in the subspace topology. As a closed subset of a compact space, A is compact. $\square$

7.2 Connectivity

Definition 7.1. A space X is *connected* if whenever $X = A \cup B$ with A, B open and $A \cap B = \varnothing$, then either $A = \varnothing$ or $B = \varnothing$.

Here are a few equivalent definitions:

1. If $X = A \cup B$ with A, B open and nonempty, then $\overline{A} \cap B = \varnothing$ or $A \cap \overline{B} = \varnothing$.

2. If $A \subseteq X$ is both open and closed, then $A = X$ or $A = \varnothing$.

3. If $A \subseteq X$ has empty boundary, then $A = X$ or $A = \varnothing$.

4. If $f : X \to \{1, 2\}$ is continuous, and $\{1, 2\}$ has the discrete topology, then f is constant.

Theorem 7.3. *$\mathbb{R}$ is connected.*

Proof. If $\mathbb{R} = A \cup B$ with A, B open and $A \cap B = \varnothing$, then $\mathbb{R} \setminus A = B$ and $\mathbb{R} \setminus B = A$ are closed. Choose $x \in A$ and $y \in B$, and assume (without loss of generality) that $x < y$. Let $X = \{b \in [x, y] : [b, y] \subseteq B\}$. We know $y \in B$ and $y \in [x, y]$, so $y \in X$, making $X \neq \varnothing$. Also, x is a lower bound for X. So $I = \inf X \in \mathbb{R}$ exists. As the infimum of X, I is a limit point of X. Since $X \subseteq B$, I is a limit point of B, so $I \in \overline{B} = B$. This means $I \notin A$. Since B is open, we can find $\varepsilon > 0$ such that $(I - \varepsilon, I + \varepsilon) \subseteq B$. So $[I - \varepsilon/2, y] \subseteq B$, contradicting the definition of I as the infimum of X. $\square$

Theorem 7.4. *A nonempty $X \subseteq \mathbb{R}$ is connected iff X is an interval (i.e. $X = (a, b)$ or $[a, b]$ or $(a, b]$ or $[a, b)$).*

Proof. ($\Longleftarrow$) This is the same proof as the previous theorem.

($\Longrightarrow$) If X is connected but X is not an interval, then there exist $a, b \in X$ and $p \in \mathbb{R} \setminus X$ such that $a < p < b$. Let $A = \{x \in X : x < p\}$, and let $B = \{x \in X : x > p\}$. Then $A, B \neq \varnothing$, as $a \in A$ and $b \in B$. We have $X = A \cup B$ and $A \cap B = \varnothing$, as $x \in X$ satisfies either $x < p$ or $x > p$. To show that A is open, we show that B is closed. Since $p \notin X$, $\overline{B} \subseteq X$ only contains points larger than p; so $\overline{B} = B$. This means that B is closed, so A is open. Similarly, A is closed, so B is open. This contradicts X being connected. $\square$

8 Connectivity, Path-Connectivity, Separation, and Metrization

8.1 Connectivity involving continuous functions and product spaces

Theorem 8.1. *If $f : X \to Y$ is continuous, and X is connected, then $f(X)$ is connected.*

Proof. For simplicity, assume $Y = f(X)$. If $Y = A \cup B$ with A, B open and $A \cap B = \varnothing$, then $X = f^{-1}(Y) = f^{-1}(A) \cup f^{-1}(B)$. We know that $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint (because $A \cap B = \varnothing$) and open (because f is continuous). X is connected, so $f^{-1}(A)$ or $f^{-1}(B)$ is $\varnothing$. Note that $f(\varnothing) = \varnothing$ and $f(f^{-1}(A)) = A$ by the surjectivity of f, so A or $B = \varnothing$. $\square$

Lemma 8.1. *If $\{A_i\}$ is a collection of connected subspaces of X, and $\cap_i A_i \neq \varnothing$, then $\bigcup_i A_i$ is connected.*

Proof. Let $p \in \bigcap_i A_i$. Suppose that $\bigcup_i A_i = B \cup C$ for open, disjoint B, C. Then $p \in B$ without loss of generality. For each A_i, $A_i = (B \cap A_i) \cup (C \cap A_i)$. These are disjoint and open (in the subspace topology). For each A_i, A_i is connected, so $B \cap A_i = \varnothing$ or $C \cap A_i = \varnothing$. But $B \cap A_i \neq \varnothing$, as it contains p. So $A_i \cap C = \varnothing$, meaning $A_i \subseteq B$. So $\bigcup_i A_i \subseteq B$, which implies that $C = \varnothing$. So $\bigcup_i A_i$ is connected. $\square$

Theorem 8.2. *X and Y are connected iff $X \times Y$ is connected.*

Proof. ($\Longleftarrow$) The projection maps $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are continuous and surjective, so by our previous theorem, X and Y are connected.

($\Longrightarrow$) If $x \in X$, then $\{x\} \times Y \cong Y$ (check this yourself). So $\{x\} \times Y$ is connected; similarly, $X \times \{y\} \cong X$ is connected for any $y \in Y$. Let $A_{x,y} = (X \times \{y\}) \cup (\{y\} \times Y)$. This is connected by our lemma, since $(X \times \{y\}) \cup (\{y\} \times Y) = \{(x, y)\} \neq \varnothing$. Fix $y_0 \in Y$. Then $X \times Y = \bigcup_{x \in X} A_{a,y_0}$, and $\bigcap_{x \in X} A_{a,y_0} = X \times \{y_0\} \neq \varnothing$. So the lemma implies that $X \times Y$ is connected. $\square$

Corollary 8.1. *$\mathbb{R}^n$ is connected.*

Proof. Use the fact that $\mathbb{R}$ is connected, and induct on n. $\square$

Corollary 8.2. *$S^n \setminus \{point\}$ is connected.*

Proof. We already showed that $\mathbb{R}^n \cong S^n \setminus \{\text{north pole}\}$. It doesn't matter which point we remove. $\square$

Corollary 8.3. *S^n is connected.*

Proof. $S^n = (S^n \setminus \{\text{north pole}\}) \cup (S^n \setminus \{\text{south pole}\})$, which are both connected and have nonempty intersection. Our lemma from before shows that S^n is connected. $\square$

8.2 Connected components

What if a space is not connected? We can try to find "maximal" connected pieces.

Definition 8.1. A *(connected) component* of a space X is a subspace $A \subseteq X$ such that A is connected, and if $A \subsetneq B$, then B is not connected.

Example 8.1. If X is connected, it has one component: the set X itself.

Example 8.2. The set $[0,1] \cup [2,3]$ has two components: $[0,1]$ and $[2,3]$.

Example 8.3. The set $[0,1) \cup (1,2]$ has two components: $[0,1)$ and $(1,2]$.

8.3 Path connectivity

Definition 8.2. A *path* in a space X is a continuous function $\gamma : [0,1] \to X$. A path is said to be a path from $\gamma(0)$ to $\gamma(1)$; here, $\gamma(0)$ is the beginning of the path, and $\gamma(1)$ is the end of the path.

Intuitively, people like to think of a "path" as the image of γ. But in fact, if we parametrize γ differently, the path may be different, even though the image will be the same (e.g. if the image is traversed more slowly with respect to t in one area).

Definition 8.3. A space X is *path-connected* if $\forall x, y \in X$ with $x \neq y$, there exists a path from x to y.

Theorem 8.3. *If X is path-connected, X is connected.*

Proof. If $X = A \cup B$ with A, B nonempty, disjoint, and open, let $x \in A$ and $y \in B$. Then let $\gamma : [0,1] \to X$ be a path from x to y. So $[0,1] = \gamma^{-1}(A) \cup \gamma^{-1}(B)$, and these are open because γ is continuous. These are also disjoint and nonempty ($0 \in \gamma^{-1}(A)$ and $1 \in \gamma^{-1}(B)$), contradicting the fact that $[0,1]$ is connected. So X is connected. $\square$

8.4 Separation and Metrization

This section won't be tested, but it is included for interest. A good reference is Munkres sections 31 to 34.

We have already discussed Hausdorff spaces. There are other types of separation axioms for topological spaces.

Definition 8.4. A topological space X is *regular* if

1. $\{x\}$ is closed for all $x \in X$.
2. For all $x \in X$ and $A \subseteq X$ closed with $x \notin A$, there exist open sets U_x and U_A with $x \in U_x$, $A \subseteq U_A$, and $U_x \cap U_A = \varnothing$.

Definition 8.5. A topological space X is *normal* if for all pairs $A, B \subseteq X$ that are closed and disjoint, there exist open sets U_A, U_B such that $A \subseteq U_A$, $B \subseteq U_B$, and $U_A \cap U_B = \varnothing$.

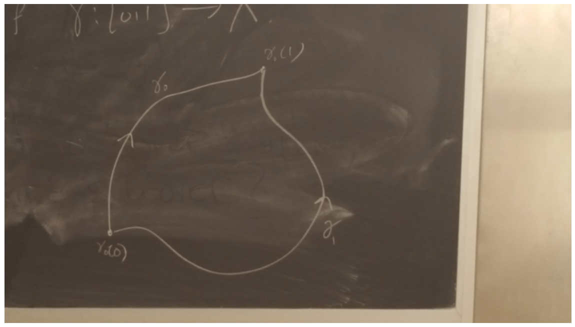
Theorem 8.4 (Urysohn's metrization lemma)**.** *If X is regular, and there exists a countable base or its topology, then X is metrizable; i.e. we can put a metric on X such that the topology induced from (X, d) is the same as the original topology.*

Remark 8.1. All metric spaces are regular, but not all metric spaces have a countable base. This second part is harder to prove.

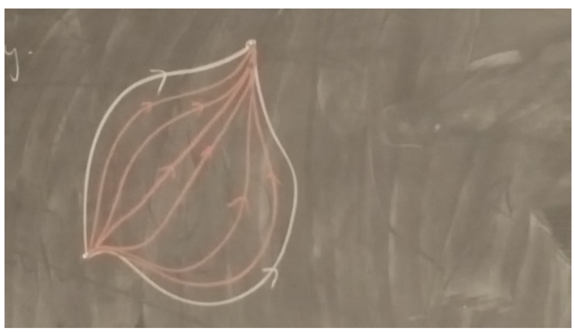
9 Homotopy

9.1 Definition and examples

Recall that a *path* is a continuous function $\gamma : [0,1] \to X$. The idea here is that we have two paths γ_0 and γ_1 in $\mathbb{R}^2$ with $\gamma_0(0) = \gamma_1(0)$ and $\gamma_0(1) = \gamma_1(1)$; i.e. the paths have the same endpoints.



Our paths are different and may even have different images, but we want to say that one can be continuously deformed into the other.



Intuitively, we want to create a family of paths $\{\gamma_t\}_{t\in[0,1]}$ "from γ_0 to γ_1." You can also say we want to interpolate continuously between γ_0 and γ_1. Think of $\{\gamma_t : t \in [0,1]\}$ as one function $\gamma : [0,1] \times [0,1] \to X$, where $\gamma(s,t) := \gamma_t(s)$.

Definition 9.1. If $f, g : X \to Y$ are continuous, then a *homotopy* F from f to g is a continuous function

$$F : X \times [0,1] \to Y,$$

where

$$F(x,0) = f(x), \qquad F(x,1) = g(x).$$

Here, we way that f is *homotopic* to g and write $f \simeq g$ (or $f \simeq_F g$).

For our paths, we want to fix the start and end, so $\gamma_t(0) = \gamma_t(1) = \gamma_0(1)$ for all $t \in [0,1]$.

Definition 9.2. If $A \subseteq X$ is a subset with $f, g : X \to Y$ continuous such that $f(a) = g(a)$ for all $a \in A$, then a *homotopy* F from f to g *relative to* A is a homotopy F from f to g such that $f(a, t) = f(a)$ for all $a \in A$. We write $f \simeq_F g$ rel A.

Example 9.1. Let f be any continuous function. Then $f \simeq f$ via $F(x, t) = f(x)$ for all $t \in [0, 1]$.

Example 9.2. If $S^1 \subseteq \mathbb{C}$ is the set $\{e^{i\theta} : \theta \in \mathbb{R}\}$ and if $\mathrm{id}_{S^1} : S^1 \to S^1$ is the identity, then $\mathrm{id}_{S^1} \simeq_F \mathrm{id}_{S^1}$, where $F(e^{i\theta,t}) = e^{i(\theta+2\pi t)}$. Here, F rotates $e^{i\theta}$ be $2\pi t$ radians, so $F(e^{i\theta}, 0) = F(e^{i\theta}, 1) = \mathrm{id}_{S'}$.

9.2 Homotopy on convex sets

Definition 9.3. A set $A \subseteq \mathbb{R}^n$ is *convex* if for all $x, y \in A$, the line segment $\{(1+t)x + ty : t \in [0, 1]\} \subseteq A$.

Example 9.3. If $Y \subseteq \mathbb{R}^n$ is convex and $f, g : X \to Y$ are continuous, then $F(x, t) = (1 - t)f(x) + tg(x)$ is a homotopy from f to g called the *straight line homotopy*.

Example 9.4. If Y is convex, $p \in Y$, $f : X \to Y$ is continuous, and $g : X \to Y$ is $g(x) = p$ for all $x \in X$, then $f \simeq g$ via the straight line homotopy.

A more general notion than a set A being convex is the notion of a set being *star-shaped*, which means that there is a point $x \in A$ such that the line segment connecting x to any $y \in A$ is contained in A.

Example 9.5. Let $X = \mathbb{R}^2 \setminus \{(0, 0)\}$, let $f : X \to X$ be the identity, and let $g : X \to X$ be $g(x) = x/\|x\|$. Then $f \simeq_F g$, where $F(x, t) = (1 - t)x + t(x/\|x\|)$. Note that X is not convex, but the line segment between x and $x/\|x\|$ is in X.

9.3 Properties of homotopy

Homotopy defines a sort of equivalence between continuous functions.

Proposition 9.1. *The relation $f \simeq g$ on the set of continuous functions form X to Y is an equivalence relation. (Similarly, $f \simeq g$ rel A is also an equivalence relation.)*

Proof. We check the three parts of the definition of an equivalence relation:

1. $f \simeq f$ by our previous example.

2. If $f \simeq_F g$, then $g \simeq_G f$, where $G(x, t) = F(x, 1 - t)$.

3. If $f \simeq_F g$ and $g \simeq_G h$, then let

$$H(x,t) = \begin{cases} F(x, 2t) & t \in [0, 1/2] \\ G(x, 2t - 1) & t \in (1/2, 1]. \end{cases}$$

   Then $f \simeq_H h$. □

Compositions of continuous functions preserve homotopy in the ways you would want.

Proposition 9.2. *If we have $f, g : X \to Y$ with $f \simeq_F g$ and $h : Y \to Z$, then $h \circ f \simeq h \circ g$. If $j : Z \to X$, then $f \circ j \simeq g \circ j$.*

Proof. For the first part, use the homotopy $h \circ F$. For the second part, use the homotopy $G(x,t) = F(j(x),t)$. □

9.4 Homotopy equivalence of spaces

Definition 9.4. If X and Y are topological spaces, then they are *homotopy equivalent* if there exist continuous functions $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. The function f is called a *homotopy equivalence* from $X \to Y$, g is its *homotopy inverse*, and we write $X \simeq Y$.

Proposition 9.3. *The relation $X \simeq Y$ is an equivalence relation.*

Proof. We check the three parts of the definition of an equivalence relation:

1. $X \simeq_{\mathrm{id}_X} X$.
2. Symmetry is built into the definition.
3. If $X \simeq_f Y$ with inverse g' and $Y \simeq_g Z$ with inverse g', then

$$\begin{aligned}(f' \circ g') \circ (g \circ f) &= f' \circ (g' \circ g) \circ f \\ &\simeq f' \circ \mathrm{id}_Y \circ f \\ &= f' \circ f \\ &\simeq \mathrm{id}_X .\end{aligned}$$

   Similarly, $(g \circ f) \circ (f' \circ g) \simeq \mathrm{id}_Z$, so $g \circ f$ is a homotopy equivalence from X to Z with homotopy inverse $f' \circ g'$. □

Definition 9.5. If $A \subseteq X$, let $i : A \to X$ be the inclusion map ($a \mapsto a$). If the map i is a homotpoy equivalence (with homotopy inverse $f : X \to X$), then we call the map $i \circ f : X \to X$ a *deformation retract* (or *deformation retraction*) of X onto A.

Example 9.6. $\mathbb{R}^2 \setminus \{(0,0)\}$ deformation retracts onto the unit circle.

Definition 9.6. If $A = \{x\} \subseteq X$, and there exists a deformation retract of X onto A, then we say that X is *contractible*.

Example 9.7. Convex subsets Y of $\mathbb{R}^n$ are contractible. If $p \in Y$, then $\mathrm{id}_X \simeq i \circ f$, where $i : \{p\} \to Y$ is the inclusion map, and $f : Y \to P$ sends $y \mapsto p$.

10 The Fundamental Group

10.1 Group structure of homotopy classes

Recall that the relation $f \simeq g$ rel A is an equivalence relation.

Definition 10.1. If X is a topological space, and $p \in X$ is a point, we define the *fundamental group of X based at p* to be the set $\pi_1(X, p)$ of homotopy classes rel $\{0, 1\}$ of continuous paths from p to p; i.e.

$$\pi_1(X, p) = \{[\gamma] : (\gamma : [0, 1] \to X) \text{ is continuous}, \gamma(0) = \gamma(1) = p\},$$

and $[\gamma] = [\gamma'] \iff \gamma \simeq \gamma'$ rel $\{0, 1\}$.

Proposition 10.1. *$\pi_1(X, p)$ is a group under the group operation $[\alpha][\beta] = [\alpha \cdot \beta]$, where $\alpha \cdot \beta : [0, 1] \to X$ is*

$$x \mapsto \begin{cases} \alpha(2x) & x \in [0, 1/2] \\ \beta(2x - 1) & x \in (1/2, 1]. \end{cases}$$

Proof. We need to check that this operation is well-defined, i.e. if $[\alpha] = [\alpha']$ and $[\beta] = [\beta']$, then $[\alpha \cdot \beta] = [\alpha' \cdot \beta']$. So if $\alpha \simeq_F \alpha'$ rel $\{0, 1\}$ and $\beta \simeq_G \beta'$ rel $\{0, 1\}$, let

$$H(x, t) = \begin{cases} F(2x, t) & x \in [0, 1/2] \\ G(2x - 1, t) & x \in (1/2, 1]. \end{cases}$$

Then $\alpha \cdot \beta \simeq_H \alpha' \cdot \beta'$ rel $\{0, 1\}$, which is what we needed.

We need to check the group axioms:

1. Closure: From the definition, we get a path from p to p.

2. Identity: Let $e : [0, 1] \to X$ be $e(x) = p$ for all $x \in [0, 1]$. Then $\alpha \cdot e$ is like α for t up to $1/2$, and then it just stays at p. We want to "slide" the $1/2$ mark over closer to 1. So let

$$F(x, t) = \begin{cases} \alpha & x \in [0, 1/2 + t/2] \\ p & x \in (1/2 + t/2, 1]. \end{cases}$$

   Then $\alpha \cdot e \simeq_f \alpha$ rel $\{0, 1\}$. Similarly, $e \cdot \alpha \simeq \alpha$ rel $\{0, 1\}$. So $[\alpha][e] = [\alpha] = [e][\alpha]$.

3. Inverses: Given a path α from p to p, let $\alpha^{-1}[0, 1] \to X$ is $x \mapsto \alpha(1 - x)$; this is running the path backwards. The idea here is that $\alpha \cdot \alpha^{-1}$ goes from p to p along α and then goes backwards; we want to start going backwards at $\alpha(1 - t)$ and then increase t. So let

$$F(x, t) = \begin{cases} \alpha((2 - t)x) & x \in [0, 1/2 + t/2] \\ \alpha^{-1}((2 - 2t)x + (2t - 1)) & x \in (1/2 + t/2, 1]. \end{cases}$$

   Then $\alpha \cdot \alpha^{-1} \simeq_F e$ rel $\{0, 1\}$. Similarly, $\alpha^{-1} \cdot \alpha \simeq_F e$ rel $\{0, 1\}$. So $[\alpha][\alpha^{-1}] = [e] = [\alpha^{-1}][\alpha]$.

4. Associativity: The idea is that $(\alpha \cdot \beta) \cdot \gamma$ acts as α and then β on the first two $1/4$ intervals and then γ on the last $1/2$; $\alpha \cdot (\beta \cdot \gamma)$ acts as α for the first $1/2$ and then β and then γ on the later two $1/4$ intervals. Instead of defining F directly, let $f : [0,1] \to [0,1]$ be
$$f(x) = \begin{cases} 2x & x \in [0, 1/4] \\ x + 1/4 & x \in (1/4 + 1/2] \\ \frac{x+1}{2} & x \in (1/2, 1]. \end{cases}$$
Then $((\alpha \cdot \beta) \cdot \gamma)(x) = (\alpha \cdot (\beta \cdot \gamma))(f(x))$. Note that since $[0,1]$ is convex, $f \simeq \mathrm{id}_{[0,1]}$ rel $\{0,1\}$ via the straight-line homotopy. So
$$\begin{aligned} (\alpha \cdot \beta) \cdot \gamma &= (\alpha \cdot (\beta \cdot \gamma)) \circ f \\ &\simeq (\alpha \cdot (\beta \cdot \gamma)) \circ \mathrm{id}_{[0,1]} \text{ rel } \{0,1\} \\ &= \alpha \cdot (\beta \cdot \gamma). \end{aligned}$$
□

10.2 Changing the basepoint

Does the fundamental group depend on the choice of point p?

Definition 10.2. $A \subseteq X$ is a *path component* if A is path-connected and for any B with $A \subsetneq B$, B is not path connected.

If $[\gamma] \in \pi_1(X, p)$, then for all $x \in [0,1]$ $\gamma(x)$ is in the same path component of X as p. A priori, $\pi_1(X, p)$ depends on p and on the path component p is in.

Theorem 10.1. *If X is path-connected, then $\pi_1(X, p) \cong \pi_1(X, q)$ for all $p, q \in X$.*

Proof. We can compose paths γ, γ' if $\gamma(1) = \gamma'(0)$. Similarly to in the previous proof,
$$(\gamma \cdot \gamma')(x) = \begin{cases} \gamma(2x) & x \in [0, 1/2] \\ \gamma'(2x - 1) & x \in (1/2, 1] \end{cases}$$
specifies a well-defined and associative operation with inverses (up to homotopy). So choose a path $\gamma : [0,1] \to X$ with $\gamma(0) = p$ and $\gamma(1) = q$. Define a map $\gamma_* : \pi_1(X, p) \to \pi_1(X, q)$ taking $[\alpha] \mapsto [\gamma^{-1} \cdot \alpha \cdot \gamma]$.

We need to check that γ_* is well-defined: this is true as composition is well-defined on homotopy classes. To check that γ_* is a homomorphism, note that
$$\begin{aligned} \gamma^{-1} \cdot (\alpha \cdot \beta) \cdot \gamma &\simeq (\gamma^{-1} \cdot \alpha) \cdot (\beta \cdot \gamma) \text{ rel } \{0,1\} \\ &= (\gamma^{-1} \cdot \alpha) \cdot e \cdot (\beta \cdot \gamma) \\ &\simeq (\gamma^{-1} \cdot \alpha) \cdot (\gamma \cdot \gamma^{-1}) \cdot (\beta \cdot \gamma) \text{ rel } \{0,1\} \\ &\simeq (\gamma^{-1} \cdot \alpha \cdot \gamma) \cdot (\gamma^{-1} \cdot \beta \cdot \gamma) \text{ rel } \{0,1\}, \end{aligned}$$
so $\gamma_*([\alpha][\beta]) = \gamma_*([\alpha])\gamma_*([\beta])$. The homomorphism γ_* is an isomorphism because it has the inverse $(\gamma^{-1})_*$. □

This allows us to write $\pi_1(X)$ for a path-connected space X.

11 Review: Identification Spaces and Embeddings

11.1 Identification spaces and continuity

11.1.1 Identification spaces and attaching maps

Let's review the concept of an identification space.

Let X be a space with a partition $\mathcal{P}$ of X. We have a function $p : X \to \mathcal{P}$ mapping x to the element of $\mathcal{P}$ containing x. Define a space Y that has:

- Points are elements of $\mathcal{P}$.
- Open sets are $U \subseteq \mathcal{P}$ such that $p^{-1}(U)$ is open.

Example 11.1. Let $X = \{1, 2, 3, 4\}$ with the open sets $\{\varnothing, \{1\}, \{3\}, \{1, 3\}, X\}$. Let

$$\mathcal{P} = \{\{1, 4\}, \{2, 3\}\}.$$

Then $p(1) = \{1, 4\}$, $p(2) = \{2, 3\}$, $p(3) = \{2, 3\}$, and $p(4) = \{1, 4\}$.

What sets are open in Y? We have $p^{-1}(\varnothing) = \varnothing \subseteq X$, so $\varnothing$ is open in Y. Similarly, $p^{-1}(\{\{1, 4\}, \{2, 3\}\}) = X$ is open, so the whole space Y is open. However, $p^{-1}(\{1, 4\}) = \{1, 4\} \subseteq X$ is not open, so $\{1, 4\}$ is not open in Y. Also, $p^{-1}(\{2, 3\}) = \{2, 3\} \subseteq X$ is not open, so $\{2, 3\}$ is not open in Y.

So we can call this space $Y = \{a, b\}$ with open sets $\{\varnothing, Y\}$, where $a = \{1, 4\}$ and $b = \{2, 3\}$.

The function $p : X \to \mathcal{P}$ corresponds to a map $p : X \to Y$. Is p continuous? If $U \subseteq Y$ is open, then $p^{-1}(U) \subseteq X$ is open; so yes, p is continuous. In general, this is not the only topology for which p is continuous, but it is the largest such topology.

Theorem 11.1. *If X is a space, Y is an identification space (created from X), and Z is another space with maps*

$$X \xrightarrow{p} Y \xrightarrow{f} Z,$$

then f is continuous iff $f \circ p$ is continuous.

Proof. This follows straight from the definitions of continuity and the topology on Y.

$$\begin{aligned} f \text{ is continuous} &\iff \forall U \subseteq Z \text{ open}, f^{-1}(U) \subseteq Y \text{ is open} \\ &\iff \forall U \subseteq Z \text{ open}, p^{-1}(f^{-1}(U)) \subseteq X \text{ is open} \\ &\iff \forall U \subseteq Z \text{ open}, (f \circ p)^{-1}(U) \subseteq X \text{ is open} \\ &\iff f \circ p \text{ is continuous.} \end{aligned}$$

□

11.1.2 The largest topology with respect to continuity

Here is question 1c from the 2016 midterm.

Let $X = \{1, 2, 3, 4, 5\}$ with the topology with the base $\{\{1\}, \{1, 2\}, \{3\}, \{4, 5\}\}$. Let $f : X \to \{a, b, c\}$ be

$$f(1) = f(3) = a, \qquad f(2) = f(4) = b, \qquad f(5) = c.$$

What is the largest topology on Y such that f is continuous?

We want $U \subseteq Y$ open iff $f^{-1}(U) \subseteq X$ open. Let's check a few sets:

- $f{-}1(\{a\}) = \{1, 3\}$ is open, so $\{a\}$ is open.
- $f^{-1}(\{c\}) = \{5\}$ is not open, so $\{c\}$ is not open.
- $f^{-1}(\{b\}) = \{2, 4\}$ is not open, so $\{b\}$ is not open.
- $f^{-1}(\{a, c\}) = \{1, 3, 5\}$ is not open, so $\{a, c\}$ is not open.
- $f^{-1}(\{b, c\}) = \{2, 4, 5\}$ is not open, so $\{b, c\}$ is not open.
- $f^{-1}(\{a, b\}) = \{1, 2, 3, 4\}$ is not open, so $\{a, b\}$ is not open.

So the largest topology on Y making f continuous is $\{\varnothing, \{a\}, Y\}$.

11.2 Embeddings

Definition 11.1. An *embedding* $f : X \to Y$ is a function such that if we consider this as a map $f : X \to f(X)$, then f is a homeomorphism. Here, $f(X)$ has the subspace topology.

Example 11.2. Let $f : \mathbb{R} \to \mathbb{R}^2$ send $x \mapsto (x, 0)$. Then f is an embedding of the real line into the plane.

Example 11.3. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then $f : \mathbb{R} \to \mathbb{R}^2$ sending $x \mapsto (x, g(x))$ is an embedding sending x to the *graph* of x.

Example 11.4. The following is not an embedding. Let $f : [0, 1) \to \mathbb{C}^2$ send $x \mapsto e^{2\pi i x}$. Here, f is a continuous bijection onto its image, the unit circle in $\mathbb{C}$. However, this is not a homeomorphism because $[0, 1/2)$ is open in the subspace topology on $[0, 1)$, but $f([0, 1/2])$ is not open in $S^1 \subseteq \mathbb{C}$.

How do we make an embedding in this case? First, let $f : [0, 1] \to \mathbb{C}$ be $f(x) = e^{2\pi i x}$. However, this is not injective, so we use an identification space. Define the partition on $[0, 1]$: $\mathcal{P} = \{\{x\} : x \neq 0, 1\} \cup \{\{0, 1\}\}$ The identification space Y is homeomorphic to S^1. We showed this in class ($B^1/S^0 \cong S^1$).

So we get an induced map $\tilde{f} : Y \to \mathbb{C}$, where $\{x\} \mapsto f(x)$, $\{0,1\} \mapsto f(0) = f(1)$, and $f(x) = \tilde{f}(p(x))$ for all $x \in [0,1]$

$$\begin{tikzcd}
{[0,1]} \arrow[r, "f"] \arrow[d, "p"] & \mathbb{C} \\
S^1 \cong Y \arrow[ur, "\tilde{f}"'] &
\end{tikzcd}$$

Here, $\tilde{f}$ is continuous iff f is continuous. We have $\tilde{f} : Y \to \mathbb{C} \cong \mathbb{R}^2$, where the domain is compact (as the continuous image of a compact space) and the codomain is Hausdorff (as a metric space), so f is a homeomorphism.

12 Induced Maps and the Fundamental Group of S^1

12.1 Induced maps

Recall that given a space X and $p \in X$, we can define the fundamental group based at p

$$\pi_1(X, p) = \{[\gamma] : (\gamma : [0,1] \to X) \text{ is continuous}, \gamma(0) = \gamma(1) = p\},$$

where $[\gamma] = [\gamma']$ iff $\gamma \simeq \gamma'$ rel $\{0,1\}$.

We also showed that the basepoint did not matter if the space was path-connected (i.e. $\pi_1(X,p) \cong \pi_1(X,q)$). The idea of this proof was that we take a path γ from p to q, and given $[\alpha] \in \pi_1(X,p)$, send $[\alpha] \mapsto [\gamma^{-1} \cdot \alpha \cdot \gamma]$. This converts paths based at p to paths based at q by running γ (and its inverse) at the beginning and end of the path. We called this isomorphism γ_*; in general this map depends on γ.

Definition 12.1. If we have a continuous map $f : X \to Y$ such that $f(p) = q$, we get a homomorphism $f_* : \pi_1(X,p) \to \pi_1(Y,q)$ sending $[\alpha] \mapsto [f \circ \alpha]$. We say f_* is *induced* by f.

The proof that f_* is a homomorphism is the same as the proof that γ_* is a homomorphism, so we will not repeat it.

Theorem 12.1. *If $f : X \to Y$ and $g : Y \to Z$, then*

$$(g \circ f)_* = g_* \circ f_*.$$

Proof. This follows from the definition and properties of compositions and homotopy. $\square$

Remark 12.1. The identitfy function $\mathrm{id}_X : X \to X$ induces an isomorphism $\pi_1(X,p) \to \pi_1(X,p)$, the identity isomorphism. So if $f : X \to Y$ is a homeomorphism, then $f_*^{-1} \circ f_* = (\mathrm{id}_X)_*$ and $f_* \circ f_*^{-1} = (\mathrm{id}_X)_*$, and we get that f_* is an isomorphism from $\pi_1(X,p) \to \pi_1(Y, f(p))$.

12.2 The fundamental groups of contractible spaces and S^1

Let's find the fundamental group of some spaces.

Example 12.1. Let X be convex (or contractible). Then $\gamma : [0,1] \to X$ with $\gamma(0) = \gamma(1) = p$ is homotopic to $\gamma_p : [0,1] \to X$ which sends $x \mapsto p$ via the straight line homotopy.[12] Recall that this is

$$F(x,t) = (1-t)\gamma(x) + t\gamma_p(x).$$

Note that $F(0,t) = (1-t)\gamma(0) + t\gamma_p(0) = p$ and $F(1,t) = p$, so $\gamma \simeq_F \gamma_P$ rel $\{0,1\}$. So $\pi_1(X,p) \cong 1$, the trivial group.

[12] In a non-convex but contractible space, you may have to use a different homotopy.

Let $S^1 = \{e^{i\theta} \in \mathbb{C}; \theta \in \mathbb{R}\}$ be the circle. Let $f : \mathbb{R} \to S'$ be $x \mapsto e^{2\pi i x}$, and let $\gamma_n : [0,1] \to \mathbb{R}$ be $x \mapsto nx$. Then $f \circ \gamma_n [0,1] \to S^1$ mapping $x \mapsto e^{2\pi i n x}$ is a path in S^1 from 1 to 1, and it wraps around the circle $|n|$ times (counterclockwise if $n > 0$ and clockwise if $n < 0$).

Theorem 12.2. *The map $\phi : \mathbb{Z} \to \pi_1(S^1, 1)$ sending $n \mapsto [f \circ \gamma_n]$ is an isomorphism.*

Proof. First note that if $\gamma'_n : [0,1] \to \mathbb{R}$ has $\gamma'_n(0) = 0$ and $\gamma'_n(1) = n$, then $\gamma_n \simeq \gamma'_n$ rel $\{0,1\}$ (as $\mathbb{R}$ is convex). This implies that $[f \circ \gamma_n] = [f \circ \gamma'_n]$.

ϕ is a homomorphism: If $m, n \in \mathbb{Z}$ let $\sigma : [0,1] \to \mathbb{R}$ send $x \mapsto \gamma_n(x) + m$. Note that $f \circ \sigma = f \circ \gamma_n$, and $\gamma_m \cdot \sigma$ is a path from 0 to $m + n$. So

$$\begin{aligned} \phi(m+n) &= [f \circ \gamma_{m+n}] = [f \circ (\gamma_m \cdot \sigma)] \\ &= [(f \circ \gamma_m) \cdot (f \circ \sigma)] \\ &= [(f \circ \gamma_m) \cdot (f \circ \gamma_n)] \\ &= [f \circ \gamma_m][f \circ \gamma_n] \end{aligned}$$

ϕ is surjective: We use a "path lifting" lemma: If σ is a path in S^1 beginning at 1, then there is a unique path $\tilde{\sigma}$ in $\mathbb{R}$ starting at 0 such that$f \circ \tilde{\sigma} = \sigma$; the map $\tilde{\sigma}$ is called a *lift* of σ. So if $\alpha \in \pi_1(S^1, 1)$, then there exists a path σ such that $\alpha = [\sigma]$. From the lemma, there exists a unique path $\tilde{\sigma} : [0,1] \to \mathbb{R}$ with $\tilde{\sigma}(0) = 0$ and $f \circ \tilde{\sigma} = \sigma$. So $[f \circ \tilde{\sigma}] = \alpha$, and then $\phi(\tilde{\sigma}(1)) = \alpha$. So ϕ is surjective.

ϕ is injective: We use a "homotopy lifting" lemma: If σ, σ' are paths from 1 to 1 in S^1 with $\sigma \simeq_F \sigma'$ rel $\{0,1\}$, then there exists a unique homotopy $\tilde{F}$ from $\tilde{\sigma}$ to $\tilde{\sigma}'$ such that $f \circ \tilde{F} = F$; here, $\tilde{F}$ is a lift of F. So if $\phi(n) = e \in \pi_1(S^1, 1)$, then $f \circ \gamma_n \simeq_F e$ rel $\{0,1\}$, where $e : [0,1] \to S^1$ sends $x \mapsto 1$. The lemma implies that there exists a unique homotopy $\tilde{F}$ such that $f \circ \tilde{F} = F$.

The domain of $\tilde{F}$ is the square $[0,1] \times [0,1]$. We also know that $F(0,t) = F(1,t) = 1$ for all $t \in [0,1]$ and that $F(x,1) = e(x) = 1$ for all $x \in [0,1]$. So if P is the union of the left, top, and right edges of the square, then $F(P) = 1$. Then $\tilde{F}(P) \subseteq \mathbb{Z}$. But P is connected, and $\mathbb{Z}$ is discrete, so $\tilde{F}(P)$ is a singleton. Observe that $\sigma(0) = \tilde{\sigma}(0) = 0$, so $\tilde{F}(0,0) = 0$; then $\tilde{F}(P) = \{0\}$. Also, $F(x,0) = \gamma_n$, as the lift is unique. So $n = \gamma_n(1) = \tilde{F}(1,0) = 0$. So $\phi(n) = e$ implies that $n = 0$, making ϕ injective. $\square$

13 Lifting Lemmas and the Fundamental Groups of S^n and T^n

Many thanks to Jiabao Yang, who provided me with his notes, since I missed this lecture.

13.1 Lifting lemmas

While proving $\pi_1(S^1, 1) \cong \mathbb{Z}$, we used two lemmas.

Lemma 13.1. *If $\sigma : [0,1] \to S^1$ is a path in S^1 starting at $1 \in S^1$, then there is a unique path $\tilde{\sigma} : [0,1] \to \mathbb{R}$ such that $\tilde{\sigma}(0) = 0$ and $f \circ \tilde{\sigma} = \sigma$, where $f : \mathbb{R} \to S^1$ is $f(x) = e^{2\pi i x}$.*

Proof. Consider an open cover $S^1 = U \cup V$, where $U = S^1 \setminus \{-1\}$ and $V = S^1 \setminus \{1\}$. Then

$$f^{-1}(U) = \mathbb{R} \setminus f^{-1}(\{-1\}) = \bigcup_{n \in \mathbb{Z}} (n - 1/2, n + 1/2),$$

where each interval $(n - 1/2, n + 1/2) \cong U$ via the homeomorphism $f|_{(n-1/2,n+1/2)}$. Similarly,

$$f^{-1}(V) = \bigcup_{n \in \mathbb{Z}} (n, n+1),$$

where each interval $(n, n+1) \cong V$ via the homeomorphism $f|_{(n,n+1)}$. Taking the path $\sigma : [0,1] \to S^1$, we have that since $\{U, V\}$ is an open cover of S^1, $\{\sigma^{-1}(U), \sigma^{-1}(V)\}$ is an open cover of $[0,1]$.

Claim: Since $[0,1]$ is a compact metric space, we can break $[0,1]$ into $[t_0, t_1] \cup [t_1, t_2] \cdots \cup [t_{m-1}, t_m]$, where $0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = 1$ and $[t_i, t_{i+1}] \subseteq \sigma^{-1}(U)$ or $\sigma^{-1}(V)$. Note that $\sigma(0) = 1 \in U$, so $[t_0, t_1] \subseteq \sigma^{-1}(U)$, or, equivalently, $\sigma([t_0, t_1]) \subseteq U$. We have a homeomorphism $(-1/2, 1/2) \to U$, so define $\tilde{\sigma}(x) = (f|_{(-1/2,1/2)})^{-1}(\sigma(x))$ for $x \in [t_0, t_1]$. Using induction, assume that $\tilde{\sigma}$ is defined on $[t_0, t_1]$. Then $\sigma([t_1, t_{i+1}]) \subseteq U$ or V.

If $\sigma([t_1, t_{i+1}]) \subseteq U$ and $\tilde{\sigma}(t_i) \in (n-1/2, n+1/2)$, define $\tilde{\sigma}(x) = (f|_{(n-1/2,n+1/2)})^{-1}(\sigma(x))$ for $x \in [t_i, t_{i+1}]$. If $\sigma([t_1, t_{i+1}]) \subseteq V$ and $\tilde{\sigma}(t_i) \in (n, n+1)$, define $\tilde{\sigma}(x) = (f|_{(n,n+1)})^{-1}(\sigma(x))$ for $x \in [t_i, t_{i+1}]$. □

Lemma 13.2. *If σ, σ' are paths from 1 to 1 in S^1 with $\sigma \simeq_F \sigma'$ rel $\{0,1\}$, then there exists a unique homotopy $\tilde{F}$ rel $\{0,1\}$ from $\tilde{\sigma}$ to $\tilde{\sigma}'$ such that $f \circ \tilde{F} = F$, where $\tilde{\sigma}, \tilde{\sigma}'$ are the lifts of σ, σ'.*

Proof. The proof of this lemma is similar to that of the previous lemma, so we just provide a sketch. Break the domain of the homotopy into small squares S_i such that $F(S_i)$ is in U or in V, and then define $\tilde{F}$ similarly to how we defined $\tilde{\sigma}$ in the previous lemma. □

For more details, see the proof of lemma 5.11 in the Armstrong textbook.

13.2 The fundamental groups of S^n and T^n

We have shown that $\pi_1(S^1) \cong \mathbb{Z}$. What about the fundamental group of S^n for $n \geq 2$?

Definition 13.1. A space X is *simply connected* if $\pi_1(X, p) \cong 1$.

Theorem 13.1. *Let X be connected, and $X = U \cup V$ open, simply connected, and path-connected. Then for $p \in U \cap V$, $\pi_1(X, p) \cong 1$.*[13]

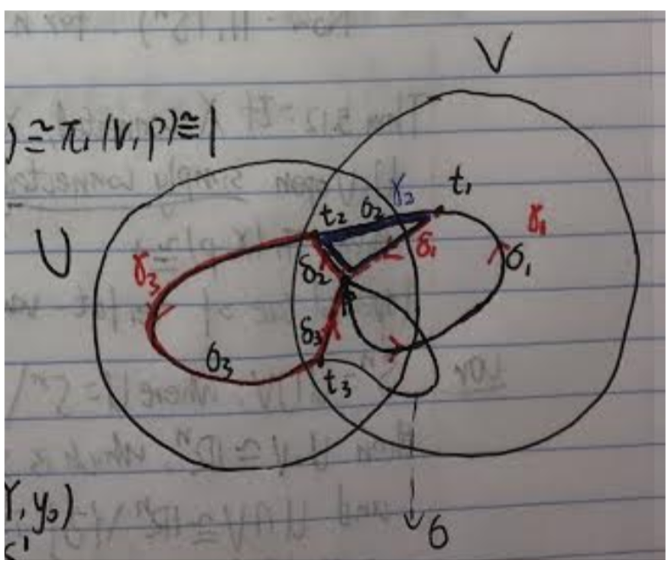
Proof. We want to show that each path σ in X from p to q is homotopic rel $\{0,1\}$ to $\gamma_1 \cdot \gamma_2 \cdot \gamma_3 \cdots \gamma_m$ for γ_i a path from p to p in U or in V. If so, then $\gamma_i \simeq e_p$ rel $\{0,1\}$ (where $e_p(x) = p$ for all x) for all $i = 1, \ldots, m$, so $\sigma \simeq e_p$ rel $\{0,1\}$. $\gamma_i \simeq e_p$ rel $\{0,1\}$ as $\pi_1(U,p) \cong \pi_1(V,p) \cong 1$.

Given $\sigma : [0,1] \to X$, choose $0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = 1$ such that $\sigma([t_i, t_{i+1}]) \subseteq U$ or V (as in the lemma before). Let σ_i be the part of σ from $\sigma(t_{i-1})$ to $\sigma(t_i)$. Let δ_i be the path from $\sigma(t_i)$ to p such that

1. δ_i is in U if $\sigma_i(t_i) \in U$,
2. δ_i is in V if $\sigma_i(t_i) \in V$,
3. δ_i is in $U \cap V$ if $\sigma_i(t_i) \in U \cap V$.

So

$$\begin{aligned}\sigma &\simeq \sigma_1 \cdot \delta_1 \cdot \delta_1^{-1} \cdot \sigma_2 \delta_2 \cdot \delta_2^{-1} \cdot \sigma_3 \cdot \delta_3 \cdots \delta_{m-1} \cdot \sigma_m \text{ rel } \{0,1\} \\ &= \gamma_1 \cdot \gamma_2 \cdots \gamma_m.\end{aligned}$$



[13] This is a special case of the Seifert-van Kampen theorem.

For each i, $[\gamma_i] \in \pi_1(U,p)$ or $\pi_1(V,p)$. But $\pi_1(U,p) \cong \pi_1(V,p) \cong 1$, so $\gamma_i \simeq e_p$ rel $\{0,1\}$. So $\sigma \simeq e_p$ rel $\{0,1\}$, and we get that $\pi_1(X,p) \cong 1$. □

Corollary 13.1. $\pi_1(S^n) \cong 1$ *for* $n \geq 2$.

Proof. $S^n = U \cup V$, where $U = S^n \setminus \{\text{north pole}\}$ and $V = S^n \setminus \{\text{south pole}\}$. Then $U, v \cong \mathbb{R}^n$, which is simply connected, and $U \cap V \cong \mathbb{R}^n \setminus \{0\}$ is path-connected for $n \geq 2$. We can then apply the theorem. □

Theorem 13.2. $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

Example 13.1. Let $T^n = S^1 \times \cdots \times S^1$ be the n-dimensional torus. Then $\pi_1(T_n) \cong \mathbb{Z}^n$.

14 Fundamental Groups of Product and Orbit Spaces

14.1 Fundamental groups of product spaces

Last time, we stated the following theorem.

Theorem 14.1. *If X, Y are topological spaces, then* $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

Proof. We have continuous maps $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$. Define $\psi : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$ as

$$[\alpha] \mapsto ((p_1)_*([\alpha]), (p_2)_*([\alpha])) = ([p_1 \circ \alpha], [p_2 \circ \alpha]).$$

Injectivity: If $p_1 \circ \alpha \simeq_F e_{x_0}$ rel $\{0, 1\}$ (where e_{x_0} is the constant path at x_0) and $p_1 \circ \alpha \simeq_F e_{x_0}$ rel $\{0, 1\}$, then $\alpha \simeq_{(F,G)} e_{(x_0,y_0)}$ rel $\{0, 1\}$. So if $\psi([\alpha]) = (e, e)$, then $[\alpha] = e$. So ψ is injective.

Surjectivity: If $[\beta] \in \pi_1(X, x_0)$ and $[\gamma \in \pi_1(Y, y_0)$, let $\alpha : [0, 1] \to X \times Y$ be $\alpha(t) = (\beta(t), \gamma(t))$ for $t \in [0, 1]$. Then $\psi([\alpha]) = ([\beta], [\gamma])$. Hence, ψ is surjective, so ψ is an isomorphism. □

14.2 Orbit spaces

14.2.1 Definitions and examples of orbit spaces

Let G be a group. (G can be thought of as a topological group with the discrete topology)

Definition 14.1. A group G *acts* on a space X if for all $g \in G$, g defines a homeomorphism $f_g : X \to X$ such that

1. For the identity $e \in G$, $f_e = \mathrm{id}_X$.

2. $\forall g, h \in G$, $f_{gh} = f_h \circ f_g$.

G acts *properly discontinuously* (called "niecly") on X if G acts on X, and $\forall x \in X$ and $g \in G$ with $g \neq e$, there exists an open neighborhood U of x such that $U \cap f_g(U) = \varnothing$.

The "nice" condition implies that if $g \neq e$, then $f_g(x) \neq x$ for each $x \in X$; i.e. there are no fixed points.

Definition 14.2. Define an identification space X/G by choosing a partition $\mathcal{P}$ on X such that x, y are in the same subset in $\mathcal{P}$ iff there exists some $g \in G$ such that $f_g(x) = y$. This identification space is called an *orbit space*.

Example 14.1. Let $X = \mathbb{R}$, and let $\mathbb{Z}$ act on $\mathbb{R}$ by $f_n(x) = x + n$. The orbit space $\mathbb{R}/\mathbb{Z} \cong S^1$, with the homeomorphism $[x] \mapsto e^{2\pi i x}$.

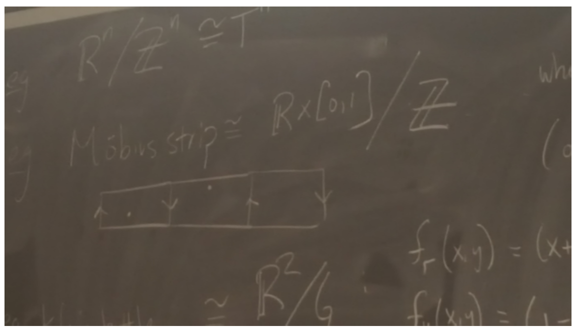
Example 14.2. Let $X = \mathbb{R}^2$, and let $\mathbb{Z}^2$ act on $\mathbb{R}$ by $f_{(m,n)}(x,y) = (x+m, y+n)$. The orbit space $\mathbb{R}^2/\mathbb{Z}^2 \cong T^2$, the torus.

This is because every $(x,y) \in \mathbb{R}^2$ is in the same equivalence class in the partition as some $(x',y') \in [0,1] \times [0,1]$. If (x,y) is in the box bounded by $x = m$, $x = m+1$, $y = n$, and $y = n+1$, then $(x' + y') = f_{(-m,-n)}(x,y)$ is in the desired unit square.

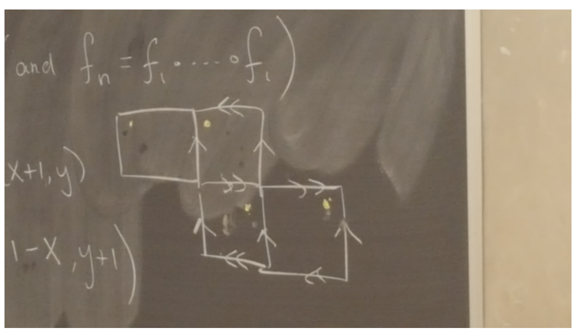
If we look at $[0,1] \times [0,1]$, the top and bottom edges get identified together by $f_{(0,1)}$, and the left and right edges get identified together by $f_{(1,0)}$. Nothing else gets identified (check this yourself), so we do indeed get the torus T^2.

Example 14.3. More generally, $\mathbb{R}^n/\mathbb{Z}^n \cong T^n$. Morally, this is because the action of $\mathbb{Z}^n$ is the product of n actions, each acting on one component of $\mathbb{R}^n$

Example 14.4. The Möbius strip is homeomorphic to $(\mathbb{R} \times [0,1])/\mathbb{Z}$, where the action is $f_1(x,y) = (x+1, 1-y)$ (and $f_n = f_1 \circ \cdots \circ f_1$ n times).



Example 14.5. The Klein bottle is homeomorphic to $\mathbb{R}^2/G$, where $G = \langle r, u \mid rur = u \rangle$, and the action is $f_r(x,y) = (x+1, y)$, and $f_u(x,y) = (1-x, y+1)$. The group elements r and u mean moving over right one square or up on square.



Example 14.6. Projective space $\mathbb{R}P^n \cong S^n/(\mathbb{Z}/2\mathbb{Z})$, where $f_1(x) = -x$.

Example 14.7. The *Lens space*[14] $L(p,q)$ for p,q relatively prime and $p > q \geq 1$ is $S^3/(\mathbb{Z}/p\mathbb{Z})$, where we think of S^3 as the unit sphere in $\mathbb{R}^4 = \mathbb{C}^2$, and $f_1(z_1, z_2) = (e^{i2\pi/p} z_1, e^{i2\pi q/p} z_2)$.

[14] Professor Conway thinks about these in his research.

Note that $e^{2\pi i/2} = -1$, so $L(2,1) \cong \mathbb{R}P^2$, so this generalizes projective space in some sense.

14.2.2 Fundamental groups of orbit spaces

Recall that simply connected means that $\pi_1(X) \cong 1$. Orbit spaces constructed from simply connected spaces have a lot of structure.

Theorem 14.2. *If G acts properly discontinuously (or "nicely") on a space X, and X is simply connected and path-connected, then $\pi_1(X/G) \cong G$.*

Proof. Let $p \in X$, and let $\pi : X \to X/G$ be the projection map (from the definition of the identification space). Let $q = \pi(p)$. If $\gamma : [0,1] \to X$ is a path from p to $f_g(p)$ (for some $g \in G$), then $(\pi \circ \gamma)(1) = \pi(\gamma(1)) = \pi(f_g(p)) = \pi(p) = q$. So $[\pi \circ \gamma] \in \pi_1(X/G, q)$.

X is simply connected, so any two such paths γ, γ' are homotopic rel $\{0,1\}$. So all we care about from γ is $\gamma(0)$ and $\gamma(1)$. Then define $\phi : G \to \pi_1(X/G, q)$ sending $g \mapsto [\pi \circ \gamma_g]$, where γ_g is a path in X from p to $f_g(p)$.

ϕ is a homomorphism: This is proved exactly like in the case $\mathbb{R} \to S^1$.

ϕ is surjective and injective: This is just like $\mathbb{R} \to S^1$, but let's give a little more description. Use:

1. Path lifting lemma: If σ is a path in X/G with $\sigma(0) = q$, there exists a unique path $\tilde{\sigma}$ in X such that $\tilde{\sigma}(0) = p$ and $\pi \circ \tilde{\sigma} = \sigma$.

2. Homotopy lifting lemma: If F is a homotopy rel $\{0,1\}$ of paths σ, σ' in X/G from q to q, then there exists a unique homotopy $\tilde{F}$ in X from the lifts $\tilde{\sigma}$ to $\tilde{\sigma}'$ (coming from path lifting) such that $\pi \circ \tilde{F} = F$.

The truth of these lemmas follows from the fact that the action is "nice." □

Corollary 14.1. $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}/2\mathbb{Z}$ *for* $n \geq 2$.

Corollary 14.2. $\pi_1($*Möbius strip*$) \cong \mathbb{Z}$.

Corollary 14.3. $\pi_1($*Klein bottle*$) \cong \langle r, u \mid rur = u \rangle$.

Corollary 14.4. $\pi_1(L(p,q)) \cong \mathbb{Z}/p\mathbb{Z}$.

15 Covering Spaces and Induced Maps of Homotopic Maps

15.1 Covering spaces

Recall from last time that if G is a group acting "nicely" on a space X, then we get an identification space X/G and a projection map $\pi : X \to X/G$. We saw that if X is path-connected and simply connected, then $\pi_1(X/G) \cong G$. Here is a new point of view:

Definition 15.1. Given a space X, a continuous function $\pi : \tilde{X} \to X$ is a *covering (space) map* and say that $\tilde{X}$ is a *covering space* (or *cover*) of X if for all $x \in X$, there exists an open neighborhood U_x of x such that $\pi^{-1}(U_x) = \bigcup_\alpha \tilde{U}_\alpha$, each $\tilde{U}_\alpha$ is open, $\tilde{U}_\alpha \cap \tilde{U}_{\alpha'} = \varnothing$, and $\pi|_{\tilde{U}_\alpha} : \tilde{U}_\alpha \to U_\alpha$ is a homeomorphism.

Example 15.1. If G is a group acting nicely on X, then $\pi : X \to X/G$ is a covering space map.

Assume X and $\tilde{X}$ are path-connected.[15] Then the same proofs as before give the following lifting lemmas.

Theorem 15.1 (path lifting)**.** *If $p \in X$ and $q \in \pi^{-1}(p)$, then every path σ in X such that $\sigma(0) = p$ has a unique lift $\tilde{\sigma}$ in $\tilde{X}$ such that $\tilde{\sigma}(0) = q$.*

Theorem 15.2 (homotopy lifting)**.** *If σ, σ' are two paths in X from p to p, and $\sigma \simeq_F \sigma'$ rel $\{0,1\}$, thene there exists a unique lift $\tilde{F}$ of F to $\tilde{X}$ such that $\tilde{\sigma} \simeq_{\tilde{F}} \tilde{\sigma}'$ rel $\{0,1\}$.*

Definition 15.2. If $\pi : \tilde{X} \to X$ is a covering space map, and $\pi^{-1}(x)$ is finite for all $x \in X$ ($|\pi^{-1}(x)| = n \in \mathbb{N}$), then we say that $\tilde{X}$ is an *n-sheeted* (or *n-fold*) covering space.

Check that if X and $\tilde{X}$ are path-connected, then this is well-defined.

Example 15.2. Let $f_n L S^1 \to S^1$ send $e^{2\pi i x} \mapsto e^{2\pi i n x}$ (where $n > 0$ is an integer). Then $f_n^{-1}(\{1\}) = \{1, e^{2\pi i/n}, e^{2\pi i(2/n)}, \ldots, e^{2\pi i(n-1)/n}\}$, so $|f_n^{-1}(1)| = n$. Check that f_n is a covering map. Then S^1 is an n-fold cover of S^1 for any $n \geq 1$.

Here, our theorem about orbit spaces doesn't apply, but $(f_n)_* : \mathbb{Z} \to \mathbb{Z}$ sending $1 \mapsto n$ is an induced homomorphism between the fundamental groups. Note that the quotient $\pi_1(S^1, 1)/(f_n)_*(\pi_1(S^1, 1)) \cong \mathbb{Z}/n\mathbb{Z}$, which has order n.

15.2 Induced maps of homotopic maps

Theorem 15.3. *If $f, g : X \to Y$ and $f \simeq_F g$, then $g_* : \pi_1(X, p) \to \pi_1(Y, g(p))$ is equal to*

$$\pi_1(X, p) \xrightarrow{f_*} \pi_1(Y, f(p)) \xrightarrow{\gamma_*} \pi_1(Y, g(p)),$$

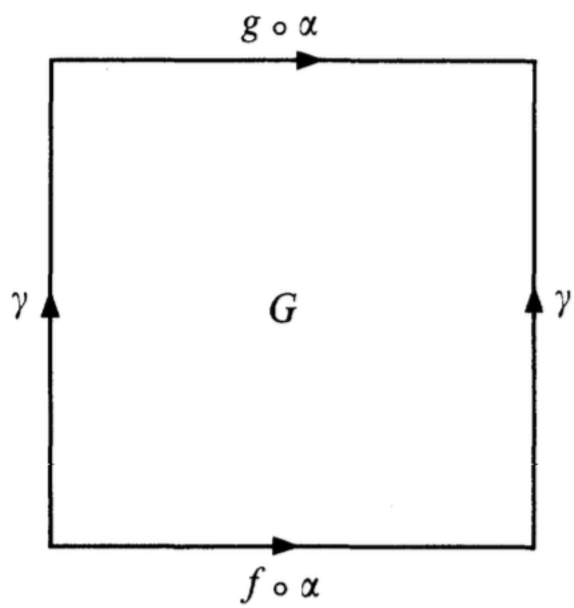
where $\gamma : [0,1] \to Y$ is the path $\gamma(x) = F(p, x)$.

[15] If $X, \tilde{X}$ are not path connected, then each component of X will have a path-connected component of $\tilde{X}$ as its covering space, so we might as well just talk about path-connected spaces.

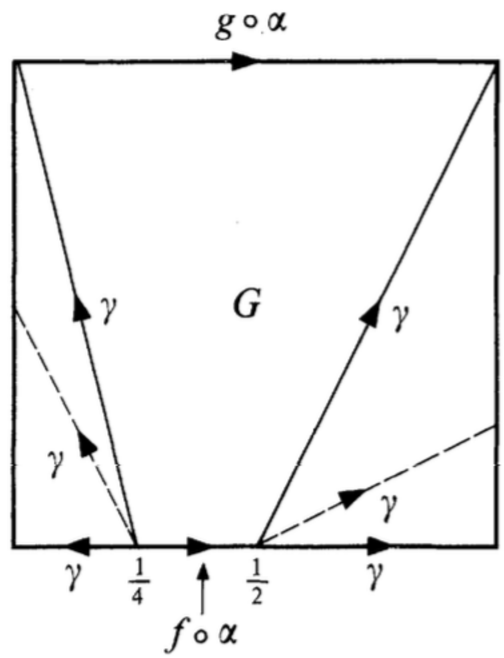
Proof. Let $\alpha : [0,1] \to X$ with $\alpha(0) = \alpha(1) = p$ be a path. Then $g_*([\alpha]) = [g \circ \alpha]$, and

$$\gamma_*(f_*([\alpha])) = \gamma_*([f \circ \alpha]) = [(\gamma^{-1} \cdot (f \circ \alpha) \cdot \gamma)].$$

We want to show that these two are equal. Let $G : [0,1] \times [0,1] \to Y$ send $(x,t) \mapsto F(\alpha(x), t)$. Drawing x on the horizontal axis and t on the vertical axis, we have the following picture for G:



Now define $H : [0,1] \times [0,1] \to Y$ according to the following picture:[16]



Then $H(x,0) = \gamma^{-1} \cdot (f \circ \alpha) \cdot \gamma$, $H(0,1) = g \circ \alpha$, $H(0,t) = \gamma(1) = g(p)$, and $H(1,t) = \gamma(1) = g(p)$. $\square$

Corollary 15.1. *If X and Y are path-connected and $X \simeq Y$, then $\pi_1(X) \cong \pi_1(Y)$.*

Proof. If $f : X \to Y$ and $g : Y \to X$ are maps such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$, then the previous theorem tells us that $(g \circ f)_* = g_* \circ f_* = \gamma_* \circ (\mathrm{id}_X)_*$ for some path γ. Then γ_* and $(\mathrm{id}_X)_*$ are isomorphisms, so $g_* \circ f_*$ is an isomorphism, as well. Since $g_* \circ f_*$ is injective, f_* is injective. Additionally, since $g_* \circ f_*$ is surjective, g_* is surjective. Similarly, $f_* \circ g_*$ is an isomorphism, so f_* is surjective, and g_* is injective. So f_* and g_* are isomorphisms. $\square$

Example 15.3. $S^1 \simeq \mathbb{R}^2 \setminus \{0\}$, the cyclinder, and the Möbius strip. So

$$\pi_1(\mathbb{R}^2 \setminus \{0\}) \cong \pi_1(\text{cylinder}) \cong \pi_1(\text{Möbius strip}) \cong \mathbb{Z}.$$

Also, the cylinder is isomorphic to $S^1 \times [0,1]$, so

$$\pi_1(\text{cylinder}) \cong \pi_1(S^1) \times \underbrace{\pi_1([0,1])}_{\cong 1} \cong \pi_1(S^1) \cong \mathbb{Z},$$

which gives us a consistent answer.

[16] An explicit formula for H is given in the proof of theorem 5.17 in the Armstrong textbook. These pictures are also taken from the Armstrong textbook.

16 The Brouwer Fixed Point Theorem and Introduction to Manifolds

16.1 The Brouwer fixed point theorem

One of the reasons why people study algebraic topology is that it can tell us things unrelated to topology itself. Here is one such theorem.

Theorem 16.1 (Brouwer)**.** *If $f : B^n \to B^n$ is continuous, then f has a fixed point; i.e. $\exists x \in B^n$ such that $f(x) = x$.*

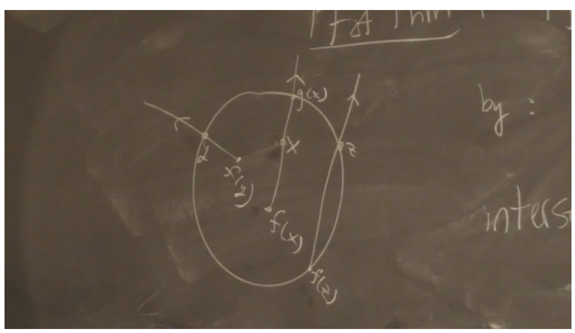
To prove this, we need the following proposition:

Proposition 16.1. *There does not exist a continuous map $f : B^n \to S^{n-1}$ such that $f(x) = x$ for all $x \in \partial B^n = S^{n-1}$.*

Proof. We did the proof of the case $n = 2$ on homeowork 6, and we will show this for $n > 3$ later. For $n = 1$, if $f : [0,1] \to \{0,1\}$ is continuous, then $f([0,1])$ is connected, so f is not surjective. $\square$

Now let's prove the fixed point theorem.

Proof. (Brouwer fixed point) Proceed by contradiction. If $f(x) \neq x$ for all $x \in B^n$, define $g : B^n \to S^{n-1}$ by drawing a ray from $f(x)$ to x and defining $g(x)$ to be the intersection of the ray with the sphere (that is not equal to $f(x)$).



Note that $g(x) = x$ for all $x \in S^{n-1}$. Check for yourself that g is a continuous function (try coming up with a formula for it). But such a g cannot exist by the previous proposition. $\square$

16.2 Introduction to Manifolds

Definition 16.1. A topological space X is *second-countable* if there exists a countable base for its topology.

Definition 16.2. A *manifold of dimension* n (or an n*-manifold*) is a topological space X such that:

1. X is Hausdorff.
2. X is second-countable.
3. $\forall x \in X$, there is an open neighborhood U_x of x and a homeomorphism $\phi : U_x \to \mathbb{R}^n$.

The pair (U_x, ϕ) is called a *chart*.

Remark 16.1. Why do we have the first two conditions? The second-countable condition excludes "weird" spaces like the "long line." The Hausdorff condition excludes spaces like the "line with 2 origins." This is $X = \mathbb{R} \cup \{0'\}$, where a set $U \subseteq X$ is open if

- $U \subseteq \mathbb{R}$, and U is open in the usual topology on $\mathbb{R}$.
- $U = (U' \setminus \{0\}) \cup \{0'\}$, where $U' \subseteq \mathbb{R}$ is open in the usual topology on $\mathbb{R}$, and $0 \in U'$.

This is second-countable, and around $x = 0'$, $(x-\varepsilon, x+\varepsilon) \cong \mathbb{R}$ and $((-\varepsilon, \varepsilon)\setminus\{0\})\cup\{0\} \cong \mathbb{R}$, so it satisfies the 3rd condition of being a manifold.

Example 16.1. $\mathbb{R}^n$ is an n-manifold.

Example 16.2. S^n is an n-manifold. If $x \in S^n$, then $S^n \setminus \{-x\}$ is an open neighborhood of x that is homeomorphic to $\mathbb{R}^n$.

Example 16.3. If X is an n-manifold, and $U \subseteq X$ is an open subspace, then U is an n-manifold.

Proposition 16.2. *If X is an n-manifold, and Y is an m-manifold, then $X \times Y$ is an $(n+m)$-manifold.*

Example 16.4. T^n is an n-manifold.

Proposition 16.3. *If X is an n-manifold, and G acts "nicely" on X, then X/G is an n-manifold.*

Proof. Given $x \in X$, let U_x be an open neighborhood of x such that $f_g(U_x) \cap U_x = \varnothing$ $\forall g \neq 0$. Then if $\pi : X \to X/G$ is the natural projection map, then $\pi|_{U_x} : U_x \to \pi(U_x)$ is a homeomorphism. The rest of the proof may be assigned for homework. $\square$

Example 16.5. $\mathbb{R}P^n$ is an n-manifold.

Example 16.6. $L(p, q)$ is a 3-manifold.

Example 16.7. The Klein bottle and the torus are 2-manifolds.

Definition 16.3. An *n-manifold with boundary* is a topological space X such that

1. X is Hausdorff.
2. X is second-countable.
3. $\forall x \in X$, there exists an open neighborhood U_x of x and a homeomorphism $\phi : U_x \to \mathbb{R}^n$ or $\phi : U_x \to \mathbb{R}^n_+$, where $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$.

The pair (U_x, ϕ) is still called a *chart.* The *interior* of X is

$$\text{int}(X) = \{x \in X : \exists \text{ chart } (U_x, \phi) \text{ s.t. } U_x \cong \mathbb{R}^n\}.$$

The *boundary* of X is

$$\partial X = X \setminus \text{int}(X) = \{x \in X : \exists \text{ chart } (U_x, \phi) \text{ s.t. } \phi(x) \in \{x_n = 0\}\}.$$

Remark 16.2. Often, authors will talk about manifolds with boundary just as "manifolds." You should always check to see which terminology is being used in whatever you are reading. There have been published results that are incorrect because they cited a result from literature without checking to make sure that the source was using the correct definition of "manifold" for their usage.

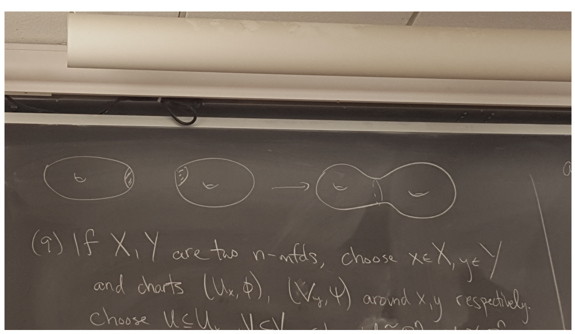
Definition 16.4. A manifold X (with boundary) is called *closed* if it is compact and $\partial X = \varnothing$.

Proposition 16.4. *If X is an n-manifold with boundary, then ∂X is an $(n-1)$-manifold.*

Example 16.8. B^n is an n-manifold with boundary, and $\partial B^n = S^{n-1}$.

Proposition 16.5. *If X, Y are two n-manifolds with boundary, and $f : \partial Y \to \partial X$ is a homeomorphism, then $X \cup_f Y$ is an n-manifold.*

Example 16.9. If X and Y are two n-manifolds, choose $x \in X$, $y \in Y$ and charts (U_x, ϕ), (V_y, ϕ). Choose $U \subseteq U_x$ and $V \subseteq V_y$ such that $U \cong_\phi B^n$ and $V \cong_\psi B^n$. Then $X' = X \setminus \text{int}(U)$ and $Y' \setminus \text{int}(V)$ are n manifolds with boundary, and $\partial X', \partial Y' \cong S^{n-1}$. Choose a homeomorphism $f : \partial Y' \to \partial X'$. Then the *connected sum* of X and Y is $X' \# Y' := X' \cup_f Y'$. (If X and Y are path-connected, then different choices of x, y, U, V, and f give homeomorphic manifolds.)



Theorem 16.2. *(Generalized Poincare ~~Conjecture~~ Theorem for topological manifolds) If X is a closed, connected n-manifold, and X is homotopy equivalent to S^n, then X is homeomorphic to S^n.*

The $n = 1, 2$ cases are "classical," and we will prove this by classifying such 1 and 2-manifolds. The $n = 3$ case was proved by Perelman in 2003, which won him a Fields medal and other prizes, all of which he rejected. The $n = 4$ case was proved by Freedman in 1982, and the $n \geq 5$ case was proven by Smale in 1960-1961.

17 Classification of 0- and 1-Manifolds

Many thanks to Jiabao Yang, who provided me with his notes, since I missed this lecture.

17.1 Classification of 0-manifolds

Theorem 17.1 (Generalized Poincarè Conjecture)**.** *If X is a closed, connected n-manifold, then $X \simeq S^n \implies X \cong S^n$.*

We will prove the cases $n = 1, 2$ by classifying such 1- and 2-manifolds.

Theorem 17.2. *All connected 0-manifolds are homeomorphic to $\{0\}$.*

Proof. If X is a 0-manifold, then for each $x \in X$, there exists and open neighborhood U_x of x and a homeomorphism $\phi : U_x \to \mathbb{R}^0$ (we called (U, ϕ) a chart). But $\mathbb{R}^0 = \{0\}$, and if $U_x \cong \{0\}$, then $U_x \cong \{x\}$. Note that this does not say that every neighborhood is one point; it says that there exists one neighborhood that is one point. So for each $x \in X$, $\{x\}$ is open, which means that X is a discrete space.[17] The connectedness of the space forces it to contain only 1 point. □

So if X is a connected, closed 0-manifold, then the statement $X \simeq S^0 \implies X \cong S^0$ vacuously holds true as there does not exist such an X such that $X \simeq S^0$.

17.2 Classification of 1-manifolds

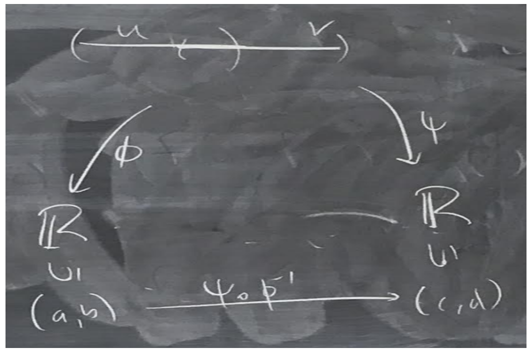
Lemma 17.1. *Let X be connected. If (U, ϕ) and (V, ψ) are charts on X and $U, V \cong \mathbb{R}$, then $U \cap V$ has at most two connected components. If $U \cap V \neq \varnothing$,*

1. *There is 1 connected component $\implies W = U \cup V \cong \mathbb{R}$.*
2. *There are 2 connected components $\implies U \cup V \cong S^1$.*

Proof. If $U \cap V \neq \varnothing$ and $U \cap V$ is connected, then $\phi(U \cap V$ and $\psi(U \cap V)$ are connected. So they are equal to (a, b) and (c, d), respectively for some $a, b, c, d \in \mathbb{R} \cup \{\pm\infty\}$ (by one of our previous theorems about connected subsets of $\mathbb{R}$). If $U \subseteq V$ or $V \subseteq U$, we are done, as $W = U$ or $W = V$. So assume neither is true, and consider $\psi \circ \phi^{-1} : (a, b) \to (c, d)$. This is

[17] In fact, every second countable (and hence countable) discrete space is a 0-manifold.

a homeomorphism. Assume $\psi \circ \phi^{-1}$ is increasing (if not, replace (U, ϕ) by $(U, -\operatorname{id}_{\mathbb{R}} \circ \phi)$).



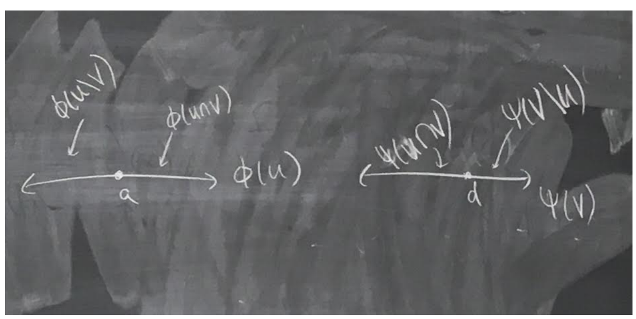
Claim: We can assume that $a \in \mathbb{R}$, $b = \infty$, $c = -\infty$, and $d \in \mathbb{R}$. If the claim is true, then assume $a < d$ (otherwise, compose ϕ with a translation). Let $f : (a, \infty) \to (a, d)$ and $g : (-\infty, d) \to (a, d)$ be homeomorphisms such that

$$(g \circ \psi)(x) = (f \circ \phi)(x) \qquad \forall x \in U \cap V.$$

Then define $\chi : U \cap V \to \mathbb{R}$ be

$$\chi(x) = \begin{cases} \phi(x) & x \in U \setminus V \\ (f \circ \phi)(x) & x \in U \cap V \\ \psi(x) & x \in V \setminus U. \end{cases}$$

Check yourself that χ is a homeomorphism.



Proof of claim: First note that $a < b \implies a \neq \infty$ and that $c < d \implies c \neq \infty$. If a, c are both finite, the consider $\tilde{a} = \phi^{-1}(a)$ and $\tilde{c} = \psi^{-1}(c)$. If $\tilde{a} \neq \tilde{c}$, then X is Hausdorff, so there exist disjoint neighborhoods $U_{\tilde{a}}, U_{\tilde{c}}$ of $\tilde{a}$ and $\tilde{c}$, respectively. So $(\psi \circ \phi^{-1})(a) = \psi(\tilde{a}) \in (c, d) \setminus \psi(U_{\tilde{c}} \cap V)$. Then $\psi \circ \phi^{-1}$ is increasing, but $(\psi \circ \phi^{-1})(a)$ is outside a neighborhood of c in (c, d). So $\psi \circ \phi^{-1}$ cannot be surjective, and $\tilde{a} = \tilde{c}$. Now $\tilde{a} = \tilde{c} \in U \cap V$, but $a = \phi(\tilde{a}) \notin \phi(U \cap V)$, which is a contradiction. So one of a, c is infinite. Similarly, only one of b, d is ∞. If $a = -\infty$ and $b = -\infty$, then $U \subseteq V$, and if $c = -\infty$ and $d = \infty$, then $V \subseteq U$. So either:

1. $a \in \mathbb{R}$, $b = \infty$, $c = \infty$, and $d \in \mathbb{R}$,

2. $a = -\infty$, $b \in \mathbb{R}$, $c \in \mathbb{R}$, and $d = \infty$.

In the second case, just switch the names of U and V. This proves the claim.

If $U \cap V$ has 2 connected components W_1 and W_2, then since U and V are connected but $U \cap V$ is not, we must have $U \not\subseteq V$ and $V \not\subseteq U$. As above,

$$\phi(W_1) = (a, b), \qquad \phi(W_2) = (a'b'),$$

$$\psi(W_1) = (c, d), \qquad \psi(W_2) = (c', d').$$

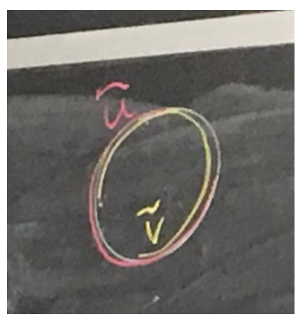
We can assume that $\phi(W_1) = (a, \infty)$ and $\phi(W_1) = (-\infty, d)$ for some $a, d, \in \mathbb{R}$. Similar analysis holds for W_2, so we conclude that

$$\phi(W_2) = (-\infty, b') \qquad \psi(W_2) = (c', \infty)$$

for some b', c' with $b' < a$ and $d < c'$. So we can write down a homeomorphism $U \cap V \to S^1$. Write $S^1 = \tilde{U} \cup \tilde{V}$, where

$$\tilde{U} = \{e^{2\pi i x} : x \in (1/4, 1)\}, \qquad \tilde{V} = \{e^{2\pi i x} : x \in (-1/4, 1/2)\}.$$

Then write a homeomorphism such that $U \to \tilde{U}$ and $V \to \tilde{V}$.



If $U \cap V$ has 3 connected components W_1, W_2, W_3, then $\phi(W_i) \subseteq \mathbb{R}$ has to be bounded for some i. But this is not possible (we skip the details due to lack of time). $\square$

We can now prove the desired classification theorem.

Theorem 17.3. *If X is a connected 1-manifold (perhaps with boundary), then X is homeomorphic to $\mathbb{R}$, S^1, $[0, 1]$, or $[0, 1)$.*

Proof. Pick a chart (U, ϕ) in X such that $\phi : U \to \mathbb{R}$ and such that (U, ϕ) is maximal; i.e. if (V, ψ) is another chart $\psi : V : \mathbb{R}$, then $U \cap V = \varnothing$ or has two components. If $X \ncong S^1$, then ant other V as above must be disjoint. If $X = U$, then $X \cong \mathbb{R}$. If not, there exists a point $p \in X \setminus U$ such that a chart (V, ψ) around p has $V \cap U \neq \varnothing$ (as X is connected). V must be homeomorphic to $\mathbb{R}^+$, and $U \cap V = V \setminus \{p\}$. If $X = U \cup \{p\}$, write a homeomorphism $X \cong [0, 1)$.

We will redo this proof next lecture, but here is the idea. If $X \ncong [0, 1)$, then $X \cong [0, 1]$; otherwise, we will get a contradiction. $\square$

18 Surfaces and Cellular Decomposition

18.1 Non-Hausdorff orbit space of a nice action

In Problem 2b on Homework 7, there was a problem that was incorrect based on the definitions we gave in class. The problem said "If X is an n-manifold with no boundary, and G acts nicely on X, then X/G is an n-manifold." It turns out that for X/G to be Hausdroff, you need additional conditions on the action.

Example 18.1. Here is a counterexample to the problem as it was written, where the orbit space X/G is not Hausdorff. Let $\mathbb{Z}$ act nicely on $X = \mathbb{R}^2 \setminus \{(0,0)\}$ by $f_n(x,y) = (2^n x, 2^{-n} y)$. Claim: In $X/\mathbb{Z}$, the images of $(1,0)$ and $(0,1)$ cannot be separated by open sets. The idea is that if you let $U_{(0,1)}$ be a small open ball around $(0,1)$ and apply f_1 repeatedly to $U_{(0,1)}$, the ball gets moved downward toward $y = 0$ and stretched wider and wider. So the image of this ball will intersect any neighborhood of $U_{(1,0)}$, and the claim holds.

18.2 Surfaces

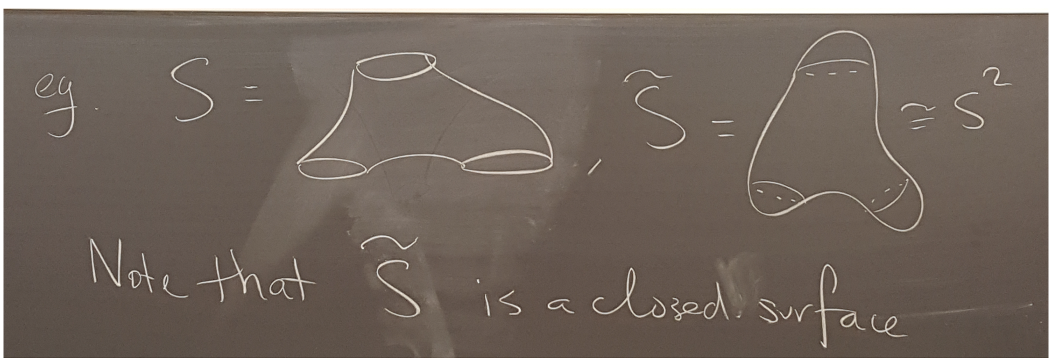
Definition 18.1. A *surface* is a 2-manifold (with or without boundary).

Proposition 18.1. *If S is a compact surface, then $\partial S \cong A_1 \amalg \cdots \amalg A_n$, where $A_i \cong S^1$ $\forall i$.*

Proof. From a proposition mentioned in class (and proved in HW7), we get that if S is a compact surface, then ∂S is a compact 1-manifold with no boundary. Then from our classification theorem, there is only one closed, connected 1-manifold, S^1. $\square$

So let $\tilde{S} = S \cup_f (D_1 \amalg \cdots \amalg D_n)$, where $D_i \cong B^2$ $(= D^2)$ for all i and the dom$(f) = \partial D_1 \amalg \cdots \amalg \partial D_n$ and $f|_{\partial D_i} : \partial D_i \to A_i$ is a homeomorphism. Note that $\tilde{S}$ is a closed surface.

Example 18.2. In the following image, S is called a "pair of pants."[18]



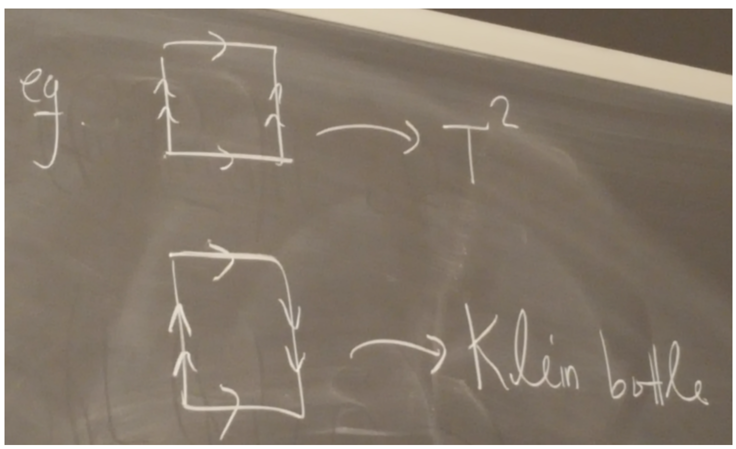
[18] Aptly named.

This means that to classify compact surfaces, we can restrict to closed surfaces.

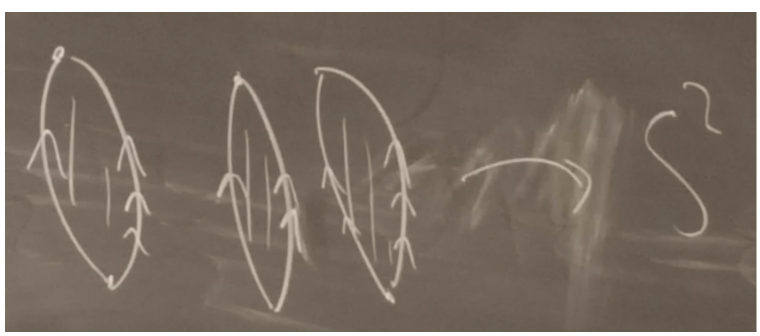
Definition 18.2. A *cellular decomposition* of a closed surface S is a collection $\{P_i, \phi_i)\}$, where $P_i \subseteq \mathbb{R}^2$ is a filled-in polygon region (e.g. a filled in pentagon) and $\phi_i : P_i \to S$ such that

1. $\forall x \in S$, $x \in \phi_i(P_i)$ for some i.
2. $\phi_i|_{\text{int}(P_i)} : \text{int}(P_i) \to S$ is an embedding.
3. $\phi_i|_{\text{int}(e)} : \text{int}(e) \to S$ is an embedding for each edge $e \subseteq P_i$.
4. If $A_{i,j} = \phi_i(P_i) \cap \phi_j(P_j) \neq \varnothing$ for some $i \neq j$, then either $\phi_i^{-1}(A_{i,j})$ and $\phi_j^{-1}(A_{i,j})$ are entire edges of P_i and P_j, or A is a singleton with $\phi_i^{-1}(A_{i,j})$ and $\phi_j^{-1}(A_{i,j})$ being vertices.

Example 18.3. Our identification space drawings with the square each constitute a cellular decomposition with a single polygon.



Example 18.4. Here is a cellular decomposition of S^2 into three bi-gons.



18.3 Outline of the classification of 2-manifolds

Here is a fact we will not prove.

Theorem 18.1. *Every surface admits a cellular decomposition. If S is compact, then it admits a finite cellular decomposition.*

We will work

Theorem 18.2. *If S is a closed, connected surface, then*

$$S \cong S^2 \# \underbrace{T^2 \# \cdots \# T^2}_{n} \# \underbrace{\mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2}_{m},$$

where n or m could be 0.

This is only half of a structure theorem. In Homework 8, we will prove the following fact.

Theorem 18.3.

$$T^2 \# \mathbb{R}P^2 \cong \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2.$$

We will prove that S^2, $T^2 \# \cdots \# T^2$, and $\mathbb{R}P^2 \# \mathbb{R}P^2$ are all distinct, which will give us our classification.

Corollary 18.1 (Classification of 2-manifolds)**.** *If S is a closed, connected surface, then either*

1. $S \cong S^2$,
2. $S \cong T^2 \# \cdots \# T^2$, *or*
3. $S \cong \mathbb{R}P^2 \# \mathbb{R}P^2$.

18.4 Single-polygon cellular decomposition and words

We will use the following:

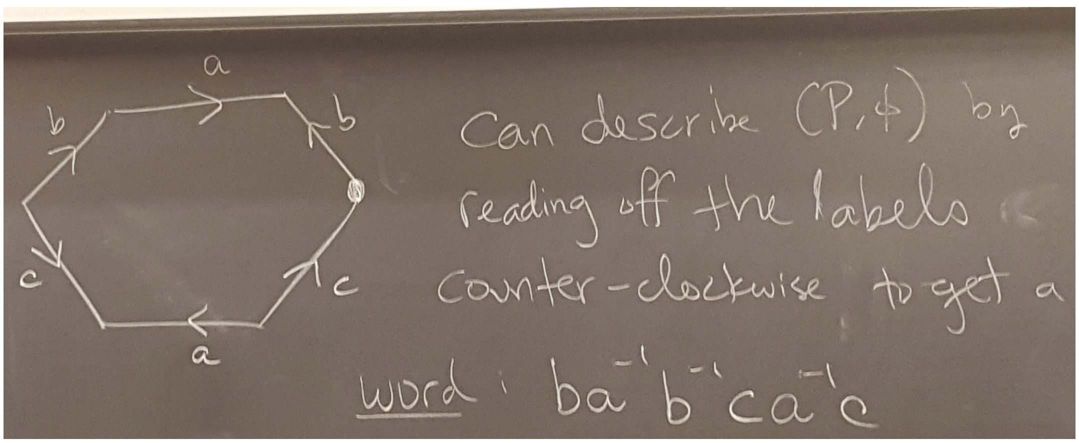
1. If $e_1 \subseteq P_1$ and $e_2 \subseteq P_2$ are edges such that $\phi_1(e_1) = \phi_2(e_2)$, then we can glue P_1 to P_2 along e_1 and e_2 to get a new polygon (P, ϕ) (where ϕ is ϕ_1 on P_1 and ϕ_2 on P_2) and replace $(P_1, \phi_1), (P_2, \phi_2)$ by (P, ϕ).
2. We can cut P_i into two pieces along a diagonal.

Lemma 18.1. *Suppose S is connected and closed. Then S has a cellular decomposition (P, ϕ) with a single polygon.*

Proof. S is compact, so choose a finite cellular decomposition $\{(P_i, \phi_i)\}$. Since S is connected and is a manifold, there exist (P_1, ϕ_1) and (P_2, ϕ_2) such that $\phi_1(e_1) = \phi_2(e_2)$ for some edges $e_1 \subseteq P_1$ and $e_2 \subseteq P_2$. Then we can glue P_1 and P_2 together along e_1 and e_2 and reduce the number of polygons in our decomposition. By repeating this process, we arrive at a single polygon. $\square$

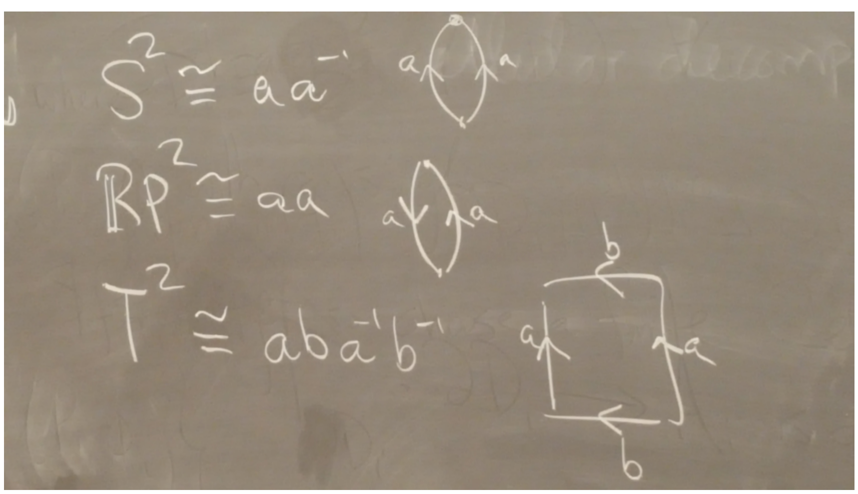
Given (P, ϕ), label the edges of P as follows: if $\phi(e_1) = \phi(e_2)$, label them the same, and put arrows indicating an orientation so that the arrows in S agree. We can describe (P, ϕ) by reading odd the labels counterclockwise to get a *word*, where the arrow gives a if the arrow goes counterclockwise and a^{-1} if the arrow goes counterclockwise. Any cyclic permutation of letters is equivalent.

Example 18.5.



We will write S as its word. So in the above example, $S \cong ba^{-1}b^{-1}ca^{-1}c$.

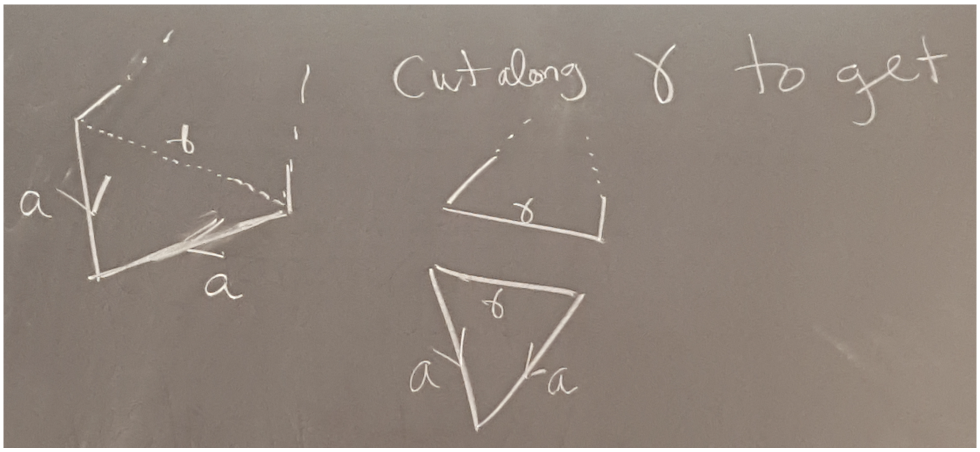
Example 18.6.



The following lemma says that adding a connecting a sphere to our surface essentially does nothing to it.

Lemma 18.2. *If $S \cong Xaa^{-1}$, and $X \neq \varnothing$, then $S \cong X$.*

Proof. We have the following picture. Cut along γ to get two polygons.



Note that $\phi(\gamma)$ is a closed loop in S, as the endpoints of γ are at vertices v_1, v_2 with $\gamma(v_1) = \gamma(v_2)$. Now glue a D^2 to each copy of γ. Notice that γ separates S into two disjoint connected components (since a only appears on one side of γ and no other letters). So $S \cong S' \# S''$. We now just need to show that one of these pieces is S^2; we will do this later. $\square$

19 Simplification of Cellular Decompositions

19.1 Removing S^2, $\mathbb{R}P^2$, or T^2

So far, we have shown that every connected, closed surface S is described by a cellular decomposition with a single polygon. We also said that a pair (P, ϕ) of a polygon and a gluing map corresponds to a word, where we read the labels of a word counterclockwise.

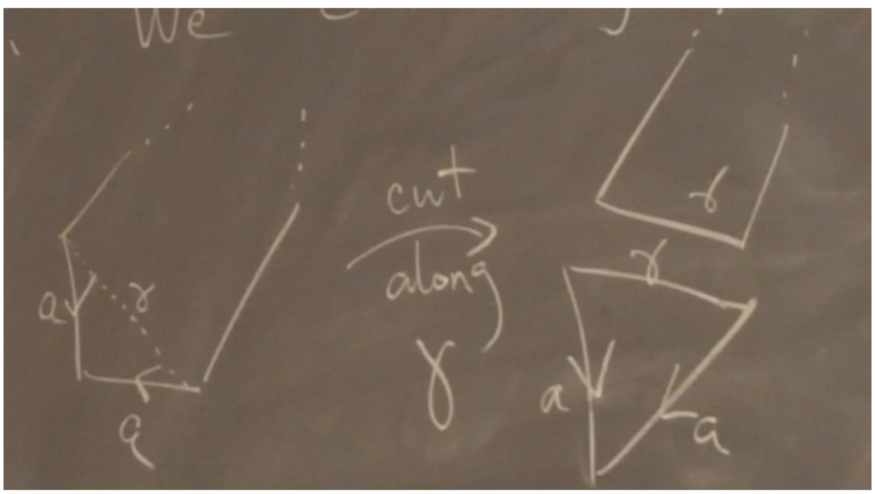
We also said that we can:

1. divide the polygon along a diagonal and reglue other edges,
2. flip a polygon over, and other homeomorphisms of $\mathbb{R}^2$.

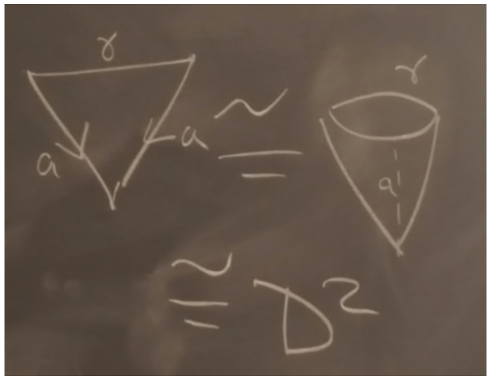
Let's continue the proof from last time.

Lemma 19.1. *If $S = Xaa^{-1}$, where $X \neq \varnothing$, then $S = X$.*

Proof. We cut along a loop γ in S.



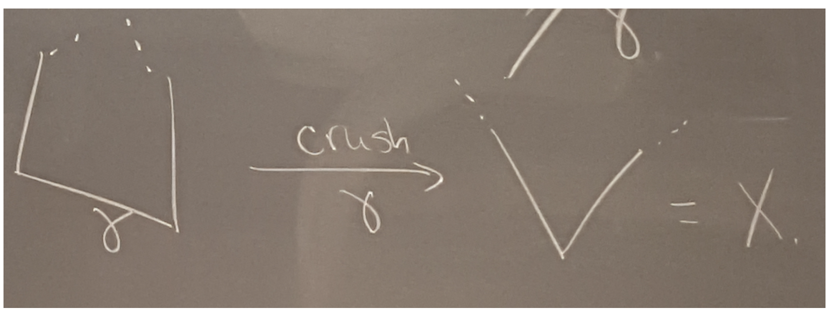
The triangle piece (call it S') is homeomorphic to a cone with circular boundary γ. This is homeomorphic to D^2.



Notice that if we glue a D^2 to $S \setminus \text{int}(S')$ and another D^2 to S', then we get:

1. from $S \setminus \text{int}(S')$: S''
2. from S': S^2,

and $S''\#S^2 \cong S$. So by HW7 problem 3, $S'' \cong S$. Glueing a disc S^2 glued along γ is homeomorphic to $(S \setminus \text{int}(S'))/\gamma$. This is



Alternatively, we can also think about it like moving in the vertex and folding the polygon in on itself.

$\square$

Lemma 19.2. *If $S = Xaa$, then $S \cong \mathbb{R}P^2$, where $S_1 = X$.*

Proof. This has the same proof as the previous lemma. $S \cong S_1 \# S_2$, where $S_2 = aa$. $\square$

Lemma 19.3. *If $S = Xaba^{-1}b^{-1}$, then $S \cong T^2 \# S_1$, where $S_1 = X$.*

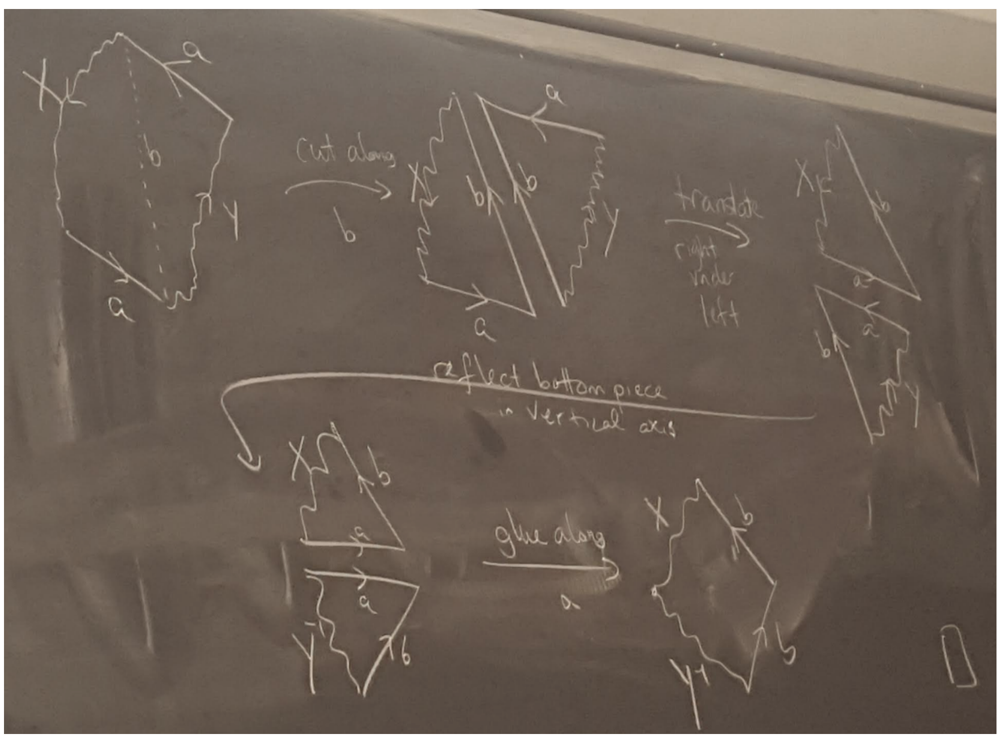
Proof. This has the same proof as, well. $S \cong S_1 \# S_2$, where $S_2 = aba^{-1}b^{-1}$. $\square$

Remark 19.1. As you can see here, we have been omitting the actual homeomorphisms. It is expected that you should be able to come up with them yourself, given some time to think about it. In general, in mathematics, it is common to omit rigorous formalism when everyone involved is expected to be familiar enough with the concepts.[19]

19.2 Rearranging and decomposition of words

Lemma 19.4. *If $S = XaYa$, and $X, Y \neq \varnothing$, then $S = bbXY^{-1}$.*

Proof. Given our polygon, cut along b to get two pieces. Then translate the right hand piece under the left, and reflect it along a vertical axis to get the a edges pointing in the same direction. Then glue it back together.

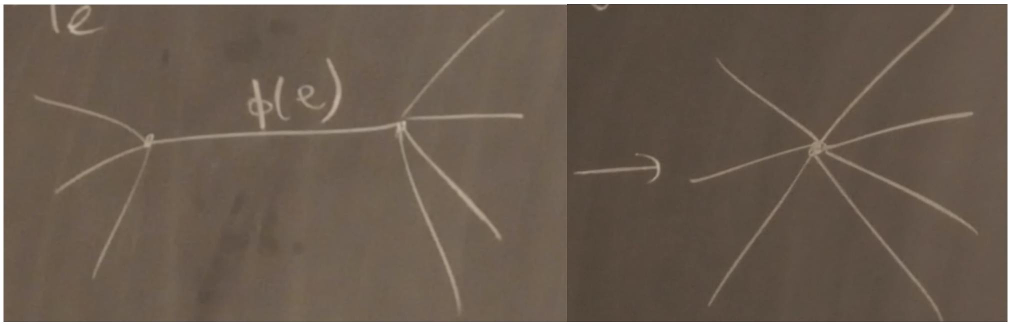


$\square$

[19] This is Professor Conway's philosophy. Personally, I prefer to always prove statements in excruciating detail.

Lemma 19.5. *We can assume that $\phi(v) = \phi(v')$ for all vertices v, v' of P.*

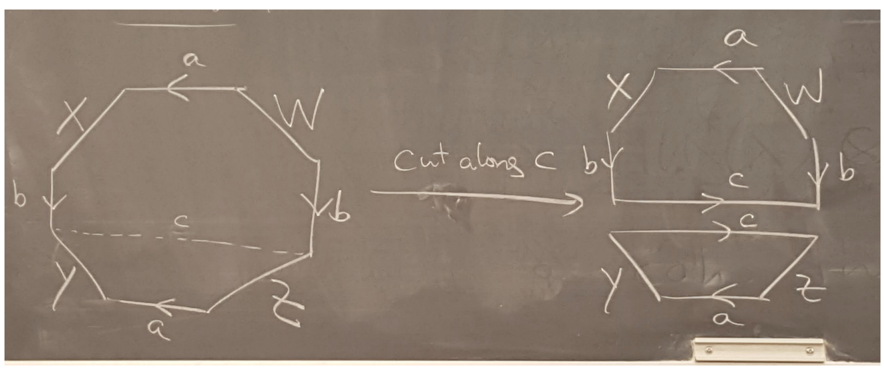
Proof. If v, v' are vertices connected by an edge e, and $\phi(v) \neq \phi(v')$, then $\phi|_e : e \to S$ is an embedding. Then we can contract $\phi(e)$ to a point.



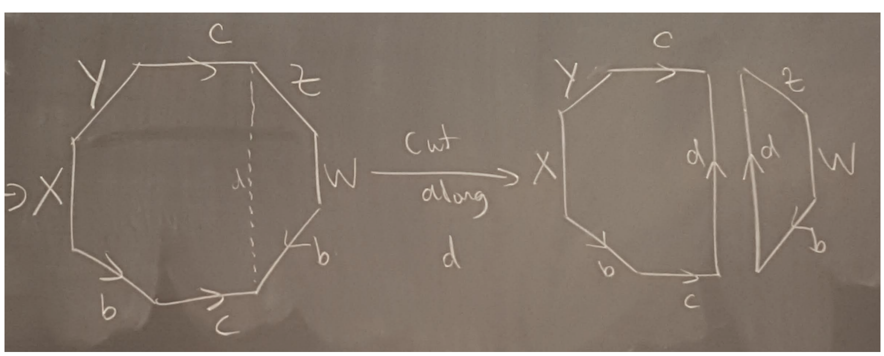
Check that we can construct (P', ϕ') from (P, ϕ) be contracting edge e and the corresponding edge e' with the same label and (P', ϕ') is a cellular decomposition of S. Then if all pairs of adjacent vertices in P satisfy $\phi(v) = \phi(v')$, then it is true for all vertices in P. $\square$

Lemma 19.6. *If $S = WaXbYa^{-1}Zb^{-1}$, then $S \cong T^2 \# S_1$, where $S_1 = ZYXW$.*

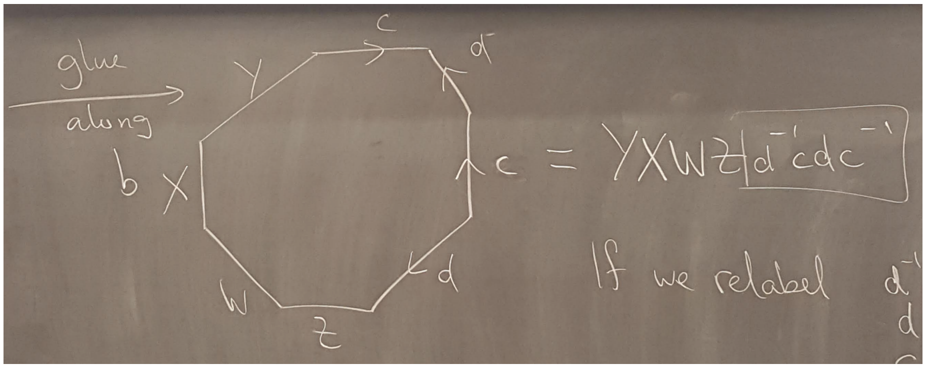
Proof. Given our polygon, cut along c to get two pieces.



Then glue along a, and then cut along d.



Then, if we glue along b, we get the word $YXWZd^{-1}cd^{-1}c$.



So our previous torus lemma gives us that $S \cong T^2 \# S_1$, where $S_1 = YXWZ = ZYXW$. $\square$

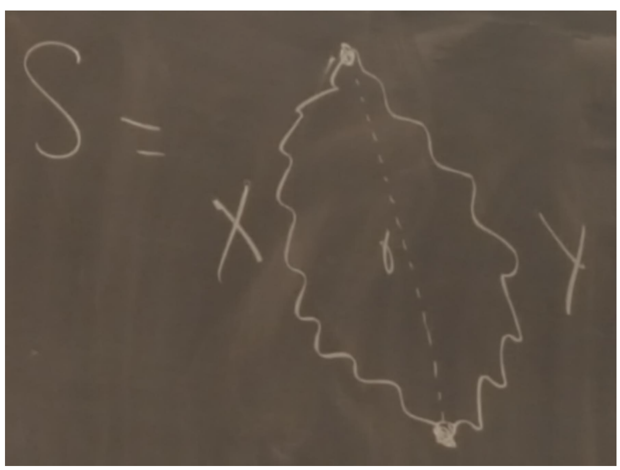
Theorem 19.1. *If S is a closed, connected surface, then*

$$S \cong S^2 \# \underbrace{T^2 \# \cdots \# T^2}_{m} \# \underbrace{\mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2}_{n},$$

where m and n can equal 0.

Proof. Anytime we see $XaYa$, use two lemmas to write $S \cong S_1 \# \mathbb{R}P^2$. By repeating this, we write $S \cong \mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2 \# S_1$, where S_1 is described by a word X such that a and a^{-1} appear for every letter (and not adjacently); note that adjacent pairs aa^{-1} give us S^2, and connect summing with S^2 does nothing (unless S^2 is the only piece).

If there are no letters a and b such that $WaXbYa^{-1}Zb^{-1}$ appears, then we have $S = XY$, where the letters in X are disjoint from the letters in Y (check this yourself). So S looks like this:



By our vertex lemma, $\phi(\gamma)$ is a closed loop. If the letters of X and Y are disjoint, then γ separates S into 2 pieces. So similar to in our sphere lemma, $S \cong S_1 \# S_2$, where $S_1 = X$ and $S_2 = Y$. So either by this argument or by our previous lemmas, we write S as the connected sum of simpler pieces (T^2 or S_1, S_2). Eventually we will run out of letters and will have S^2 or T^2. $\square$

20 Free Products and the Seifert-van Kampen Theorem

20.1 Free products and free groups

So far, we have proven the following "almost-classification."

Theorem 20.1. *If S is a closed surface, then*

$$S \cong S^2, \qquad S \cong T^2 \# \cdots \# T^2, \qquad \text{or} \qquad S \cong \mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2.$$

We want to prove that these are all distinct. Let's give these names.

Definition 20.1. For $g \in \mathbb{N}$, let

$$S_g := \underbrace{T^2 \# \cdots \# T^2}_{g}, \qquad N_g := \underbrace{\mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2}_{g}.$$

We call g the *genus* of the surface.

We will prove that genus is well-defined by showing that S^2, S_g, and N_g are all different. The idea is to calculate $\pi_1(S)$ and show that these are different for these surfaces. We know that:

$$\pi_1(S^2) \cong 1, \qquad \pi_1(T^2) = \mathbb{Z}^2$$
$$\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}^2, \qquad \pi_1(K) = \pi_1(N_2) \cong \langle r, u \mid rur = u \rangle .$$

First, let's review some group theory. We can generate a group by a presentation, which includes generators and relations between them.

Example 20.1. Here is a group with two generators and one relation.

$$\left\langle a_1, a_2 \mid a_1 a_2 a_1^{-1} a_2^{-1} = 1 \right\rangle \cong \mathbb{Z}^2.$$

Definition 20.2. Let $G = \langle a_1, \dots, a_n \mid r_1 = 1, \dots, r_m = 1 \rangle$ and $G' = \langle b_1, \dots b_{n'} \mid s_1 = 1, \dots, s_{m'} = 1 \rangle$ be finitely generated groups. Then the *free product* of G and G' is

$$G * G' = \langle a_1, \dots, a_n, b_1, \dots b_{n'} \mid r_1 = 1, \dots, r_m = 1, s_1 = 1, \dots, s_{m'} = 1 \rangle .$$

Definition 20.3. The *free group on n generators* is the group $F_n = \langle a_1, \dots, a_n \rangle$ (no relations).

The free group on 1 generator is $F_1 \cong \mathbb{Z}$. By induction, we see that the free group on n generators is $F_n \cong F_{n-1} * \mathbb{Z} \cong \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n}$.

20.2 The Seifert-van Kampen theorem

Recall a theorem we proved earlier.

Theorem 20.2. *If $X = A \cup B$ with A and B open, simply connected, and path-connected and $A \cap B$ path-connected, then $\pi_1(X) \cong 1$.*

This is a special case of a more general result.

Theorem 20.3 (Seifert-van Kampen[20])**.** *Let $X = A \cup B$ with A and B open and path-connected, $p \in A \cap B$, $A \cap B$ be path-connected, and let*

$$i_A : A \cap B \to A, \qquad i_B : A \cap B \to B$$

be the inclusion maps. Then

$$\pi_1(X, p) \cong \frac{\pi_1(A, p) * \pi_1(B, p)}{N},$$

where N is the smallest normal subgroup containing the elements $(i_A)_*(g)[(i_B)_*(g)]^{-1}$ for all $g \in \pi_1(A \cap B, p)$.

The reason we want to quotient out by this subgroup is that we want to say that $(i_A)_*(g)[(i_B)_*(g)]^{-1}$ is trivial in $\pi_1(X, p)$. That is, $(i_A)_*(g) = (i_B)_*(g)$. We have to manually insert this relation because the free product of G and G' does not include any relations relating elements of G to elements of G'.

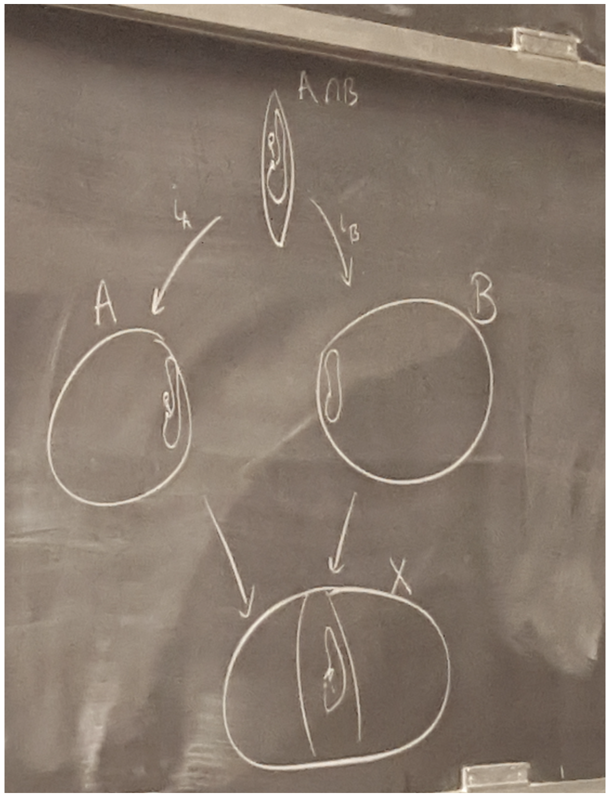
So if

$$\begin{aligned} \pi_1(A, p) &= \langle a_1, \dots, a_n \mid r_1 = 1, \dots, r_m = 1 \rangle, \\ \pi_1(B, p) &= \langle b_1, \dots b_{n'} \mid s_1 = 1, \dots, s_{m'} = 1 \rangle \\ \pi_1(C \cap B, p) &= \langle g_1, \dots g_\ell \mid t_1 = 1, \dots, t_k = 1 \rangle, \end{aligned}$$

then

$$\begin{aligned} \pi_1(X, p) = \langle a_1, \dots, a_n, b_1, \dots b_{n'} \mid & r_1 = 1, \dots, r_m = 1, s_1 = 1, \dots, s_{m'} = 1, \\ & (i_A)_*(g_1) = (i_B)_*(g_1), \dots, (i_A)_*(g_\ell) = (i_B)_*(g_\ell) \rangle. \end{aligned}$$

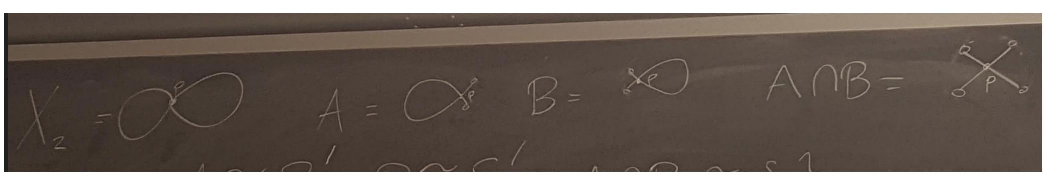
[20] This was apparently proven independently by both Seifert and van-Kampen. Sometimes, it is just called the van Kampen theorem.



20.3 Applications of the Seifert-van Kampen theorem

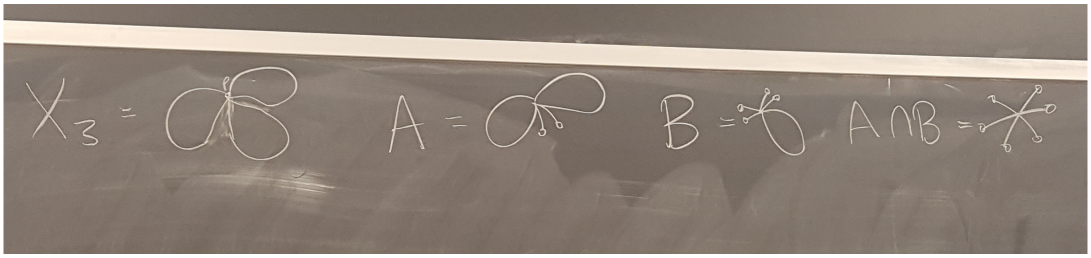
We will not prove the Seifert-van Kampen theorem, but here are some examples.

Example 20.2. Let X_2 be the 1 point union of two circles, and split into A and B as follows.



Then $A \simeq S^1$, $B \simeq S^1$, and $A \cap B \simeq \{p\}$. Since $\pi_1(A \cap B) \cong 1$, the normal subgroup $N = 1$. So $\pi_1(X_2, p) \cong \pi_1(A, p) * \pi_1(B, p) \cong \mathbb{Z} * \mathbb{Z} \cong F_2 = \{a_1, a_2\}$. The element $a_i = [\sigma_i]$, where σ_i is a path from p to p that goes once around the i-th circle.

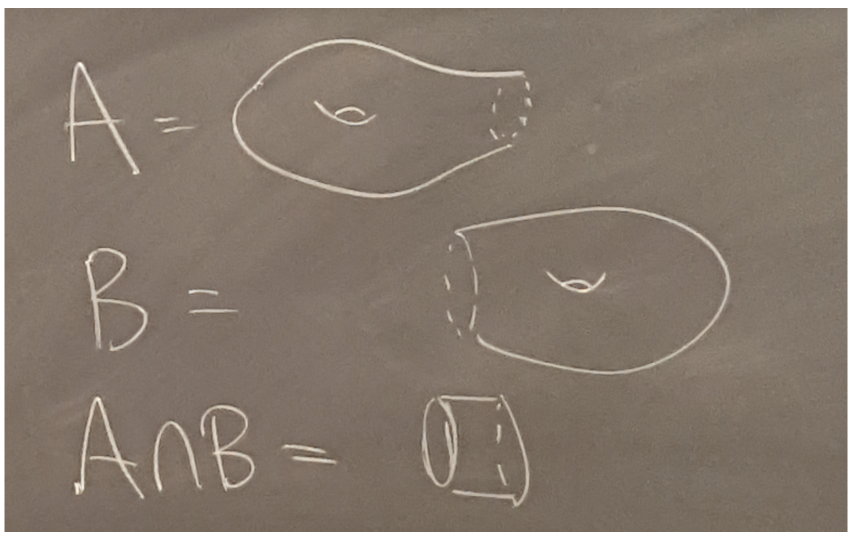
Let X_3 be the 1 point union of three circles, and split into A and B as follows.



We know that $A \simeq S_2$, $B \simeq S^1$, and $A \cap B \simeq \{p\}$. As before, $\pi_1(A \cap B, p) \cong 1$, so $N = 1$. So $\pi_1(X_3, p) \cong \pi_1(X_2) * \pi_1(S^1) \cong F_2 * \mathbb{Z} \cong F_3$.

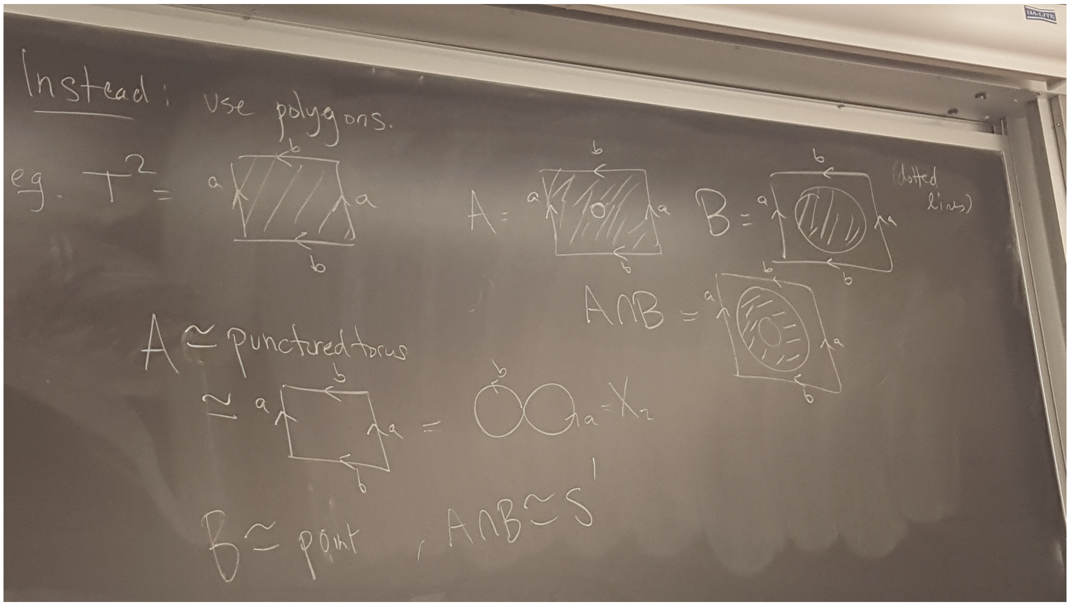
Similarly, by induction, if X_n is the 1 point union of n circles, then $\pi_1(X_n, p) \cong F_n = \langle a_1, \ldots, a_n \rangle$, and $a_i = [\sigma_i]$, where σ_i is a path p to p that goes around the i-th circle once.

Example 20.3. We can form $X = S_2$ from two punctured tori.



This can be a bit confusing, so for surfaces, we will instead use polygons.

Example 20.4. Let's decompose the torus into a punctured torus and a disc.



As we did on a homework, A deformation retracts to the edges (by widening the hole), which is actually the one-point union of two circles. B deformation retracts to a single point, and $A \cap B \simeq S^1$. We have that

$$(i_A)_* : \underbrace{\pi_1(A \cap B)}_{\cong \mathbb{Z}} \to \underbrace{\pi_1(B)}_{\cong 1} \qquad \text{sends } n \mapsto 1,$$

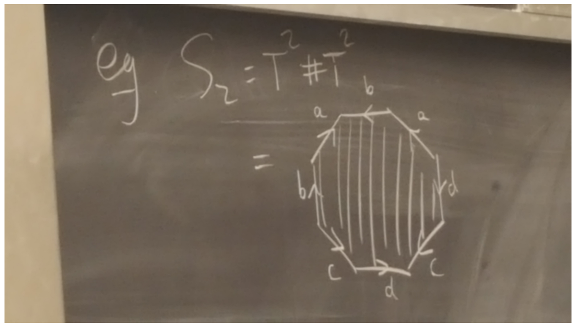
$$(i_B)_* : \underbrace{\pi_1(A \cap B)}_{\cong \mathbb{Z}} \to \underbrace{\pi_1(A)}_{\cong \langle a,b \rangle} \qquad \text{sends } 1 \mapsto aba^{-1}b^{-1},$$

which goes counterclockwise around the square. So

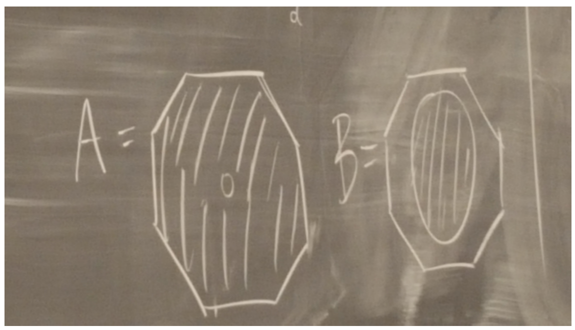
$$\begin{aligned}
\pi_1(T^2) &\cong \frac{\pi_1(A) * \pi_1(B)}{N} \\
&\cong \langle a, b \mid (i_A)_*(1) = (i_B)_*(1) \rangle \\
&= \left\langle a, b \mid aba^{-1}b^{-1} = 1 \right\rangle \\
&= \langle a, b \mid ab = ba \rangle \\
&\cong \mathbb{Z}^2.
\end{aligned}$$

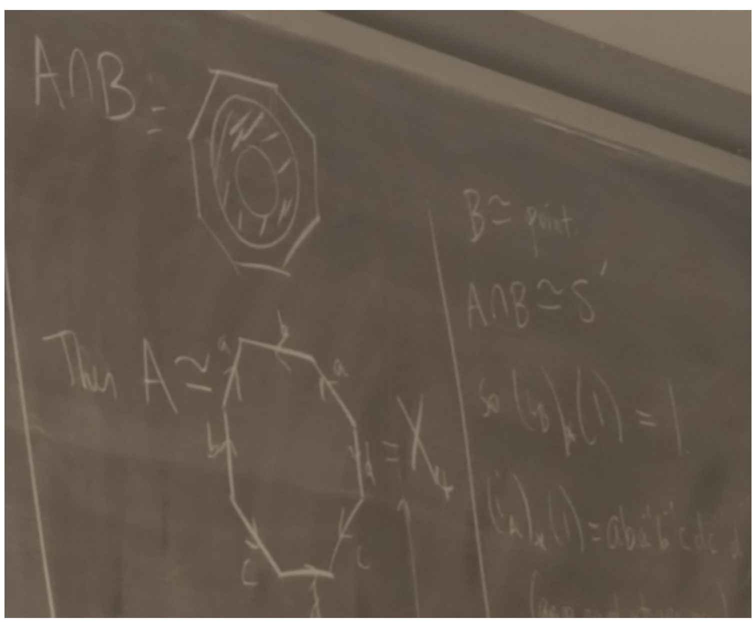
This is the third way we have calculated $\pi_1(T^2)$. The first was that we treated T^2 as $S^1 \times S^1$, and the second was that we treated T^2 as the orbit space $\mathbb{R}^2/\mathbb{Z}^2$.

Example 20.5. Look at $S_2 = T^2 \# T^2$. The single-cell cellular decomposition for S_2 is



Define A and B similarly to how we did for the torus.



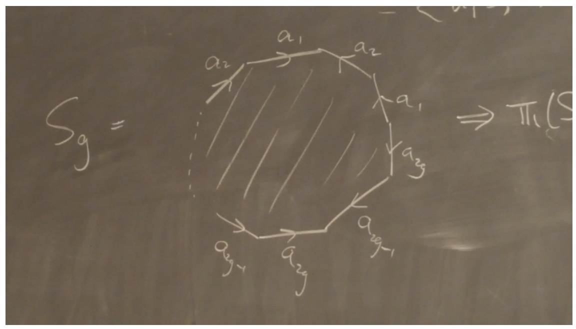


Then A deformation retracts onto the edge, which is X_4, the one-point union of 4 circles. B deformation retracts to a point, and $A \cap B \simeq S^1$. So $(i_B)_*(1) = 1$, and $(i_A)_*(1) = aba^{-1}b^{-1}cdc^{-1}d^{-1}$ (going around the octagon once). So

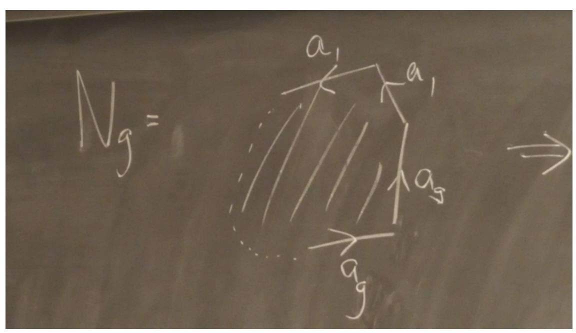
$$\begin{aligned}\pi_1(S_2) &\cong \frac{\langle a, b, c, d\rangle * 1}{N} \\ &\cong \langle a, b, c, d \mid (i_A)_*(1) = (i_B)_*(1)\rangle \\ &\cong \left\langle a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1\right\rangle,\end{aligned}$$

which is not a group we recognize. In general, we can get

$$\pi_1(S_g) \cong \langle a_1, \ldots, a_{2g} \mid a_1a_2a_1^{-1}a_2^{-1} \cdots a_{2g-1}a_{2g}a_{2g-1}^{-1}a_{2g}^{-1} = 1\rangle.$$



Example 20.6. We can do the same thing with N_g.



We get that

$$\pi_1(N_g) \cong \left\langle a_1, \ldots, a_g \mid a_1^2a_2^2 \cdots a_g^2 = 1\right\rangle.$$

How do we know if any of these groups are the same? We will abelianize them.

21 Abelianization and Other Algebraic Topology Topics

Many thanks to Jiabao Yang, who provided me with his notes, since I missed this lecture.

21.1 Abelianization

Last lecture, we encountered a problem: groups can be complicated, and we can't tell whether they are different or not based on group presentations.[21] The solution, in our case, is that abelian groups are not as complicated.

Definition 21.1. If G is a group, then its *Abelianization* $\mathrm{Ab}(G)$ is G/N, where N is the smallest normal subgroup of G containing $g_1 g_2 g_1^{-1} g_2^{-1}$ for all $g_1, g_2 \in G$.

If $G = \langle a_1, \dots, a_n \mid r_1 = 1, \dots, r_m = 1 \rangle$, then we add $n(n-1)/2$ relations to get

$$\mathrm{Ab}(G) = \langle a_1, \dots, a_n \mid r_1 = 1, \dots, r_m = 1, a_1 a_2 = a_2 a_1, a_1 a_3 = a_3 a_1, \dots, a_{n-1} a_n = a_n a_{n-1} \rangle .$$

Example 21.1.

$$\mathrm{Ab}(F_2) = \mathrm{Ab}(\langle a_1, a_2 \rangle) = \langle a_1, a_2 \mid a_1 a_2 = a_2 a_1 \rangle \cong \mathbb{Z}^2.$$

Here is a fact we will not prove.

Theorem 21.1. *IF $G \cong G'$, then $\mathrm{Ab}(G) \cong \mathrm{Ab}(G')$.*

The converse is not true, however.

Example 21.2. $F_2 \ncong \mathbb{Z}^2$, but $\mathrm{Ab}(F_2) \cong \mathbb{Z}^2 \cong \mathrm{Ab}(\mathbb{Z}^2)$.

Example 21.3. Let A_5 be the alternating group on five elements. This is nontrivial, but $\mathrm{Ab}(A_5) \cong 1$.

Here is another fact we will not prove.

Proposition 21.1. *If $r_i = 1$ is a relation in G, then any permutation of the letters of r_i is an equivalent relation in $\mathrm{Ab}(G)$.*

Example 21.4. Let $G = \big\langle a, b \mid abab^{-1} = 1 \big\rangle$. Then

$$\begin{aligned}
\mathrm{Ab}(G) &= \big\langle ab \mid abab^{-1} = 1, ab = ba \big\rangle \\
&= \big\langle ab \mid aabb^{-1} = 1, ab = ba \big\rangle \\
&= \big\langle ab \mid a^2 = 1, ab = ba \big\rangle \\
&\cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}.
\end{aligned}$$

So $\mathrm{Ab}(\big\langle a, b \mid abab^{-1} = 1 \big\rangle) \cong \mathrm{Ab}(\big\langle a, b \mid a^2 = 1 \big\rangle)$.

[21] In general, this problem is undecidable.

So if we reorder the group before abelianization, we get the same group (up to isomorphism) after abelianization.

Example 21.5.

$$\begin{aligned}\mathrm{Ab}(\pi_1(S_g)) &= \mathrm{Ab}(\langle a_1, \dots, a_{2g} \mid a_1 a_2 a_1^{-1} a_2^{-1} \cdots a_{2g-1} a_{2g} a_{2g-1}^{-1} a_2 g^{-1} = 1\rangle) \\ &\cong \mathrm{Ab}(\langle a_1, \dots, a_{2g} \mid a_1 a_1^{-1} a_2 a_2^{-1} \cdots a_{2g-1} a_{2g-1}^{-1} a_{2g} a_2 g^{-1} = 1\rangle)\end{aligned}$$

This relation just becomes $1 = 1$, so we can ignore it.

$$\begin{aligned}&= \mathrm{Ab}(\langle a_1, \dots, a_{2g}\rangle) \\ &= \mathbb{Z}^{2g}.\end{aligned}$$

So $\pi_1(S_g)$ for different g are different, as after abelianization, the $\mathrm{Ab}(\pi_1(S_g))$ are not isomorphic for different g.

Example 21.6.

$$\mathrm{Ab}(\pi_1(N_g)) = \mathrm{Ab}(\big\langle a_1, \dots, a_g \mid a_1^2 a_2^2 \cdots a_g^2 = 1\big\rangle) \cong \mathbb{Z}^{g-1} \times \mathbb{Z}/2\mathbb{Z},$$

where if $\mathbb{Z}^{g-1} \times \mathbb{Z}/2\mathbb{Z} = \mathrm{Ab}(\big\langle b_1, \dots, b_g \mid b_g^2 = 1\big\rangle)$, then the isomorphism sends $b_i \mapsto a_i$ for $i = 1, \dots, g-1$ and $b_g \mapsto a_1 a_2 \cdots a_g$ (check this yourself).

Since $\mathrm{Ab}(\pi_1)$ is distinct for every surface in our list, we conclude that no two of $S^2, S_1, S_2, \dots, N_1, N_2, \dots$ are homeomorphic. So we have proved the Poincarè conjecture for $n = 2$!

21.2 Miscellaneous topics in algebraic topology

The rest of this lecture is non-testable material but is included for interest.

21.2.1 Euler characteristic and orientibility

Here are two more things about surfaces:

1. Euler characteristics: $\chi(S) =$ "# vertices" - "# edges" + "# polygons" in a cellular decomposition. Check that all operations won't change this invariant.
2. orientibility: does a Möbius band embed in your surface (chapter 7 in Armstrong)

Using these two ideas, we can classify surfaces without using fundamental groups.

21.2.2 Homology

Definition 21.2. If X is a path-connected topological space, the *first homology group* of X is $H_1(X) = \mathrm{Ab}(\pi_1(X))$.

If X is not necessarily path-connected, $H_0(X) \cong \mathbb{Z}^{\#\text{path-components}}$.

21.2.3 Low and high dimensional topology

In general, we can classify the study of manifolds by their dimension:

- $n \leq 3$: low dimensional topology (not enough room to go wrong, not weird)
- $n = 4$: most weird things happen (enough room to go wrong, not enough room to fix them)
- $n \geq 5$: high dimensional topology[22] (enough room to go wrong, enough room to fix them)

21.2.4 Higher homotopy groups

Choose $1 \in S^1$ and $p \in X$. Then

$$\pi_1(X,p) = \{\text{homotopy classes rel } \{1\} \text{ of continuous maps } S^1 \to X \text{ s.t. } 1 \mapsto p\}.$$

Definition 21.3. Let $x_0 \in S^n$ and $p \in X$. The *n-th homotopy group* of X based at p is

$$\pi_n(X,p) = \{\text{homotopy classes rel } \{x_0\} \text{ of continuous maps } S^n \to X \text{ s.t. } 1 \mapsto p\}.$$

What is the group operation? First, let $S^n \cong B^n \setminus \partial B^n$ and $B^n \cong \underbrace{[0,1] \times \cdots \times [0,1]}_{I^n}$. So a map $f : S^n \to X$ such that $f(x_0) = p$ can be thought of as a map

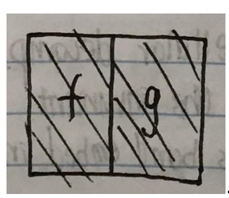
$$I^n \xrightarrow{\phi} B^n \xrightarrow{p} S^n \xrightarrow{f} X.$$

We have a projection map $p : B^n \to S^n$, and we can let $x_0 = p(\partial B^n)$. The map $|phi : I^n B^n$ is a homeomorphism.

Now, given $f, g \in \pi_n(X,p)$, let $f \cdot g \in \pi_n(X,p)$ be

$$(f \cdot g)(x_1, \ldots, x_n) = \begin{cases} f(2x_1, x_2, \ldots, x_n) & x_1 \in [0, 1/2] \\ g(2x_1 - 1, x_2, \ldots, x_n) & x_1 \in (1/2, 1] \end{cases}$$

Example 21.7. Let $n = 2$. Then this looks like



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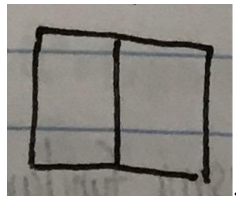
[22] This is Professor Conway's area of research.

and $f \cdot g$ evaluates to p on the set



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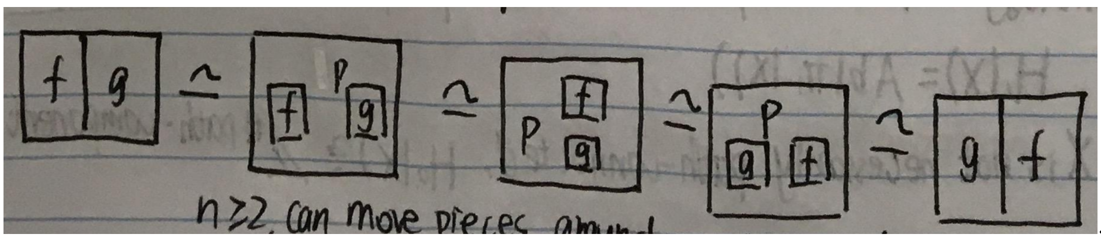
Theorem 21.2. *$(\pi_n(X,p), \cdot)$ is an Abelian group for all $n \geq 2$.*

Proof. Here is an intuitive sketch of why this is true. For $n \geq 2$, we can moe pieces around in a different direction.



, $\square$

Theorem 21.3. *Paths from p to q induce isomorphisms $\pi_n(X,p)$ to $\pi_n(X,q)$, so if X is path-connected, we can write $\pi_n(X)$.*

Theorem 21.4. *For all $n > 1$, $\pi_n(S^1) \cong 1$.*

Proof. Here is the idea. Let $g : S^n \to S^1$. Find the lift $\tilde{g}$ of g. $\mathbb{R}$ is contractible, so $\tilde{g}$ is null homotopic. So g is, as well. $\square$

Theorem 21.5. *For all $i < n$, $\pi_i(S^n) \cong 1$.*

Proof. Here is the idea. Show that any $g : S^i \to S^n$ is homotopic to $h : S^i \to S^n$ and $S^n \setminus h(S^i) \neq \varnothing$. Choose q in the complement. Then h is really a map $S^i \to S^n \setminus \{q\} \cong \mathbb{R}^n$. $\mathbb{R}^n$ is contractible, so h (and hence g) is null homotopic. $\square$

Theorem 21.6. *$\pi_n(S^n) \cong \mathbb{Z}$ and is generated by $[\mathrm{id}_{S^n}]$.*

We can use this ro prove Brouwer's fixed point theorem in all dimensions. Homology is an easier way to do so.

What about $\pi_n(S^k)$ for $n > k > 1$? This is HARD. For the last 60 years, algebraic topologies have tried to solve this; now people are bored.

Example 21.8.

$$\pi_3(S^2) \cong \mathbb{Z},$$
$$\pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z},$$
$$\pi_{14}(S^4) \cong \mathbb{Z}/120\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

There seems to be no general formula, but there exist techniques and subtle patterns, such as $\pi_{n+1}(S^n) \cong \mathbb{Z}/2\mathbb{Z}$ for all $n \geq 3$.

22 Review: Free Products, Surfaces, and Orbit Spaces

22.1 Free products vs Cartesian products

What is the difference between $\mathbb{Z}^2$ and F_2? $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$, while $F_2 = \mathbb{Z} * \mathbb{Z}$. The difference is that in $\mathbb{Z}^2$, we assume that elements commute, while they do not in F_2. If we write out the presentations, we have

$$\mathbb{Z} = \langle a \rangle, \qquad F_2 = \langle a, b \rangle.$$

The elements of F_2 are $a, b, a^{-1}b, aba^2b^3a^{-7}b^4, \ldots$.

$$\mathbb{Z}^2 = \langle a, b \mid ab = ba \rangle.$$

The elements of $\mathbb{Z}^2$ are $a, b, a^{-1}b, a^{-4}b^8, \ldots$, noting now that a and b commute. So we have elements of the form $a^m b^n$ for $n, m \in \mathbb{Z}$.

Here is one of the practice problems for Midterm 2. It says to compute the Abelianization of the following group.

$$G = \left\langle a, b, c \mid ab^2a^{-1} = 1, ac^{-1} = 1 \right\rangle$$

The second relation says that $a = c$, so we can replace all instances of c by a.

$$= \left\langle a, b \mid ab^2a^{-1} = 1 \right\rangle$$

The remaining relation says that $ab^2 = a$, which then simplifies to $b^2 = 1$.

$$\begin{aligned} &= \left\langle a, b \mid b^2 = 1 \right\rangle \\ &= \underbrace{\mathbb{Z}}_{a} * \underbrace{\mathbb{Z}/2\mathbb{Z}}_{b} \end{aligned}$$

So then

$$\mathrm{Ab}(G) = \left\langle a, b \mid b^2 = 1, ab = ba \right\rangle \cong \mathbb{Z} \times \mathbb{Z}_2.$$

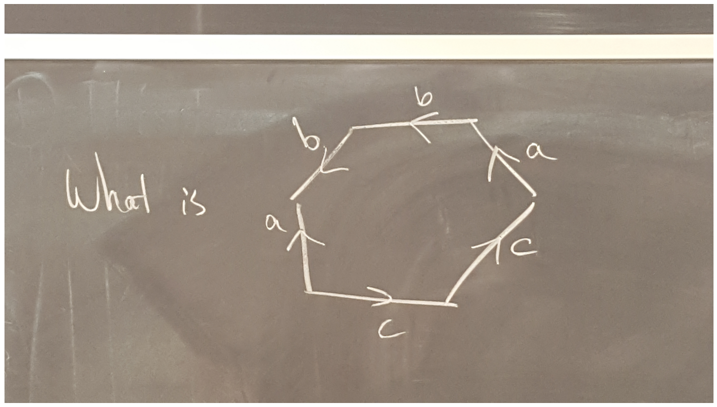
This generalizes to the following fact.

Theorem 22.1.

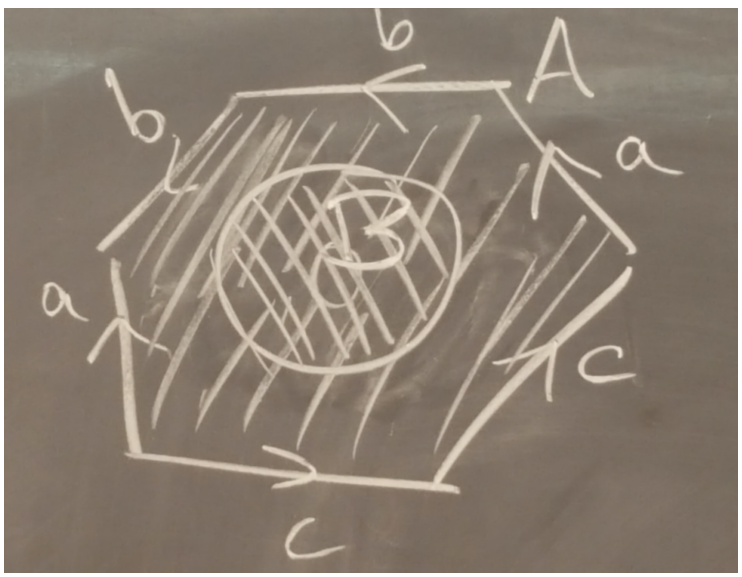
$$\mathrm{Ab}(G_1 * G_2) \cong \mathrm{Ab}(G_1) \times \mathrm{Ab}(G_2).$$

22.2 Recognizing surfaces using the Seifert-van Kampen theorem

How can we tell what a surface is given a cellular decomposition?



Our first way to do this is to use our lemmas about the word of a cellular decomposition. Another is to use the Seifert-van Kampen theorem to separate the surface into parts.

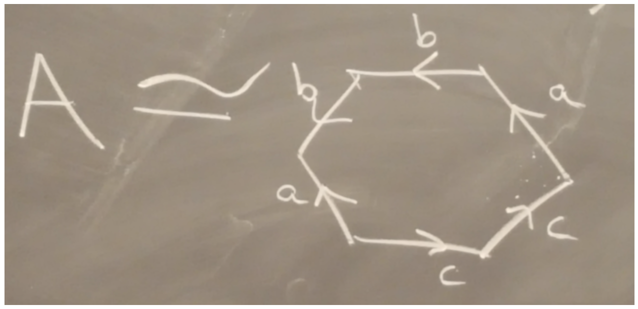


$$B \cong D^2 \simeq \text{point}$$

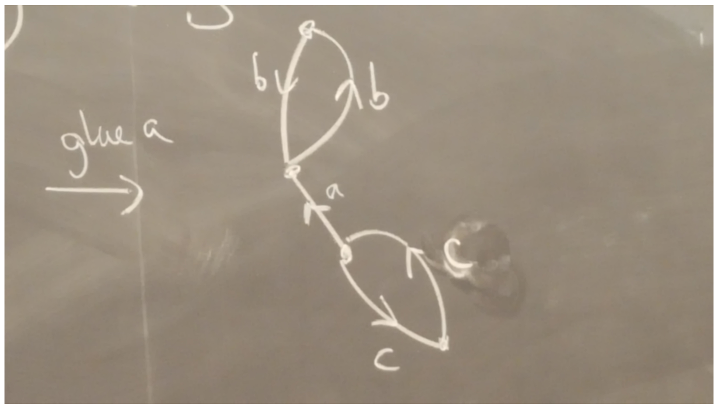
$$A \cap B \cong S^1 \times (0,1) \simeq S^1$$

$$A \simeq H,$$

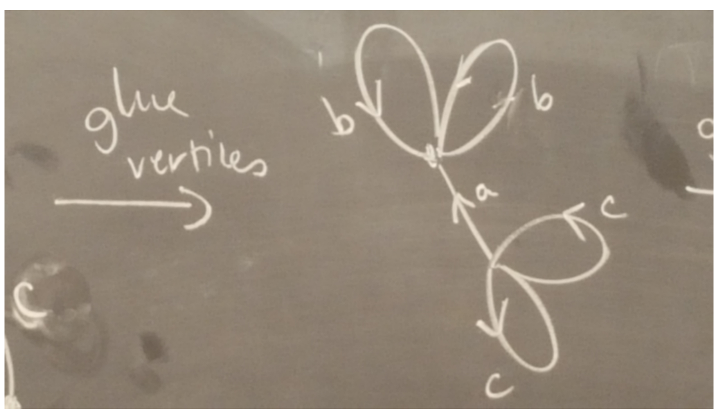
where H is the boundary of this hexagon.



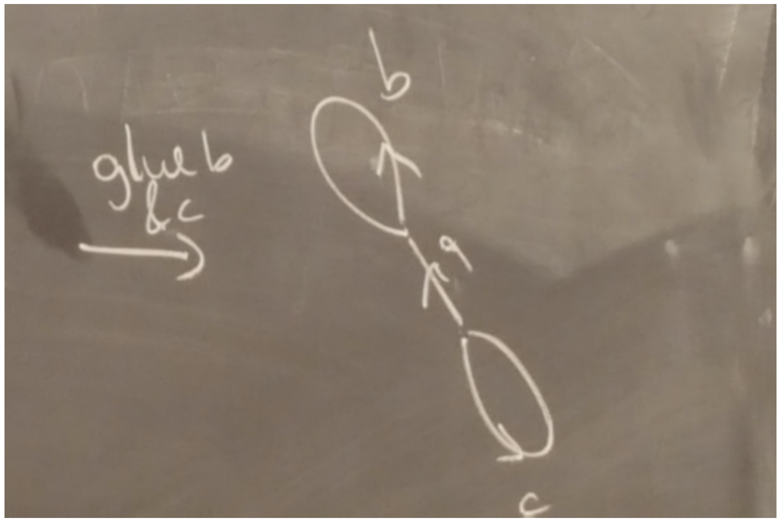
To figure out what H is homotopy equivalent to, glue the a sides together.



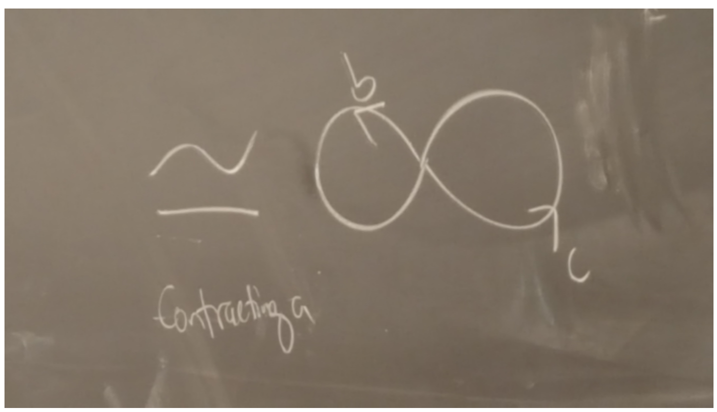
The two vertices of the b edges are the same, and the two vertices of the c edges are the same, so glue them together.



Then we can overlap the b loops together and the c loops together to get



Shortening the side labeled a gives us



So we get that

$$\pi_1(A \cap B) \cong \mathbb{Z} = \{[\gamma]^n : n \in \mathbb{Z}\}, \qquad \pi_1(A) \cong F_2 = \langle b, c \rangle, \qquad \pi_1(B) \cong 1.$$

If $i_1 : A \cap B \to A$ and $i_B : A \cap B \to B$ are the inclusion maps, then

$$\pi_1(X) \cong \frac{\pi_1(A) * \pi_1(B)}{N} \cong F_2/N = \langle b, c \mid (i_A)_*([\gamma]) = (i_B)_*([\gamma]) \rangle$$

Note that $i_A([\gamma])$ is going once around the border of the hexagon. So, looking at our pictures, we get that this is going around loop b twice and then loop c twice.

$$\pi_1(X) \cong \left\langle b, c \mid b^2 c^2 = 1 \right\rangle.$$

What group is this?

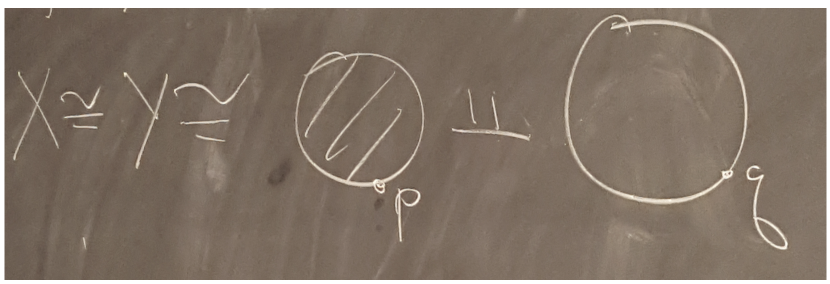
$$\begin{aligned}
\mathrm{Ab}(\pi_1(X)) &= \big\langle b, c \mid b^2c^2 = 1, bc = cb \big\rangle \\
&= \big\langle b, c \mid (bc)^2 = 1, bc = cb \big\rangle \\
&= \big\langle bc, c \mid (bc)^2 = 1, (bc)c = c(bc) \big\rangle \\
&= \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}.
\end{aligned}$$

So by our classification theorem for surfaces, $X \cong \mathbb{R}P^2 \# \mathbb{R}P^2$.

22.3 Homotopy equivalence and the fundamental group

If $X \simeq Y$, then is $\pi_1(X, p) \cong \pi_1(Y, q)$? This only holds true in general when X and Y are path-connected. You have to make sure that p and q are on the same connected component.

Example 22.1. Here is an example where the statement does not hold. Let $X \cong Y \cong D^2 \amalg S^1$.



Then $\pi_1(X, p) \cong 1$, but $\pi_1(Y, q) \cong \mathbb{Z}$.

However, taking care with the basepoints, we do have the following theorem.

Theorem 22.2. *If $f : X \to Y$ is a homotopy equivalence, then $\pi_1(X, p) \cong \pi_1(Y, f(p))$.*

22.4 Covering spaces and orbit spaces

Here is Problem 3b from the 2016 midterm: "Give a covering space of $\mathbb{R}P^n$."

The easiest answer to give is $\mathbb{R}P^n$ itself because $X \xrightarrow{\mathrm{id}_X} X$ is a covering map. We could also have $\mathbb{R}P^n \amalg \cdots \amalg \mathbb{R}P^n$.

If we want a nontrivial, path-connected covering space, we should use S^n. There is an action of $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ on S^n given by

$$f_0 = \mathrm{id}_{S^n}, \qquad f_1(x) = -x.$$

Then $\mathbb{R}P^n \cong S^n/(\mathbb{Z}/2\mathbb{Z})$ under this action. To show that the action is nice, take the U to be the interior of a hemisphere (say, the upper hemisphere) containing x; then $f_1(U)$ is the lower hemisphere, which is disjoint.

We also had the following theorem to help us figure out the fundamental group of $\mathbb{R}P^n$.

Theorem 22.3. *If $\pi_1(X) \cong 1$ and G acts nicely on X, then $\pi_1(X/G) \cong G$.*

Here some of the orbit spaces we talked about:

$$\mathbb{R}P^n \cong S^n/(\mathbb{Z}/2\mathbb{Z}), \qquad T^n \cong \mathbb{R}^n/\mathbb{Z}^n$$

For the midterm, you should also know about $B^n = D^n$, S^n, $\mathbb{R}^n$, the surfaces S_g, and N_g, the Klein bottle, and the Möbius strip.

Why doesn't the Möbius strip M deformation retract onto $\partial M \cong S^1$? $\pi_1(M) \cong \mathbb{Z}$, and $\pi_1(\partial M) \cong \pi_1(S^1) \cong \mathbb{Z}$. However, if $i : \partial M \to M$ were a homotopy equivalence, then $i_* : \pi_1(\partial M) \to \pi_1(M)$ is an isomorphism. Check that i_* is multiplication by 2 (or -2) and therefore can't be an isomorphism.

23 Knot Theory

23.1 Knot equivalence and isotopy

We have been using arguments with less rigor for two reasons.

1. The details are very similar to things we've already done the details for.
2. We have limited time to tackle as many interesting concepts as we can.

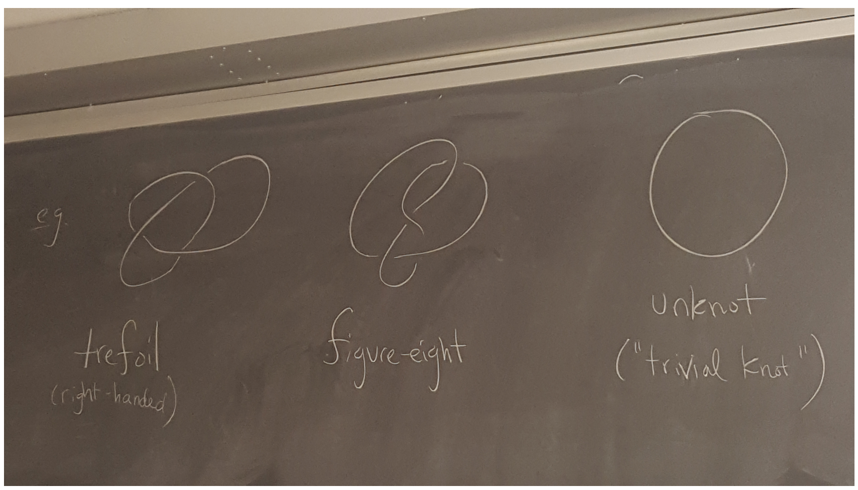
We may have to sacrifice rigor here, as well, for the same reasons.

Definition 23.1. A *knot* is a subspace of $\mathbb{R}^3$ that is homeomorphic to S^1.

Definition 23.2. A *link* is a subspace of $\mathbb{R}^3$ that is homeomorphic to $S^1 \amalg \cdots \amalg S^1$.

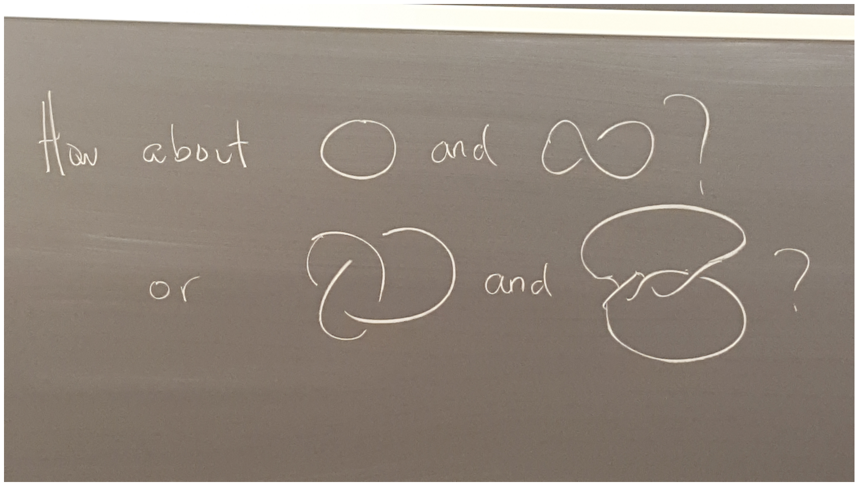
We can't draw in 3D, so we need to draw projections of knots to the plane and keep track of over/under crossings.

Example 23.1.



What about the difference between these knots?[23]

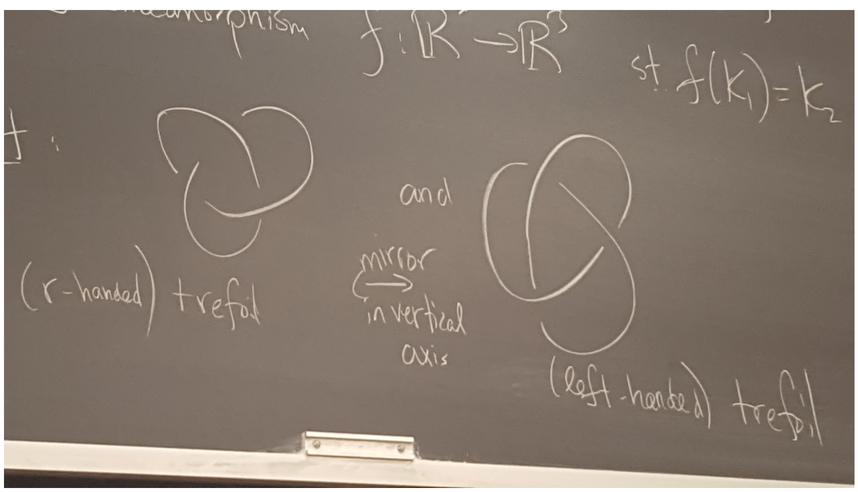
[23] Knot drawing is an acquired skill. There is a video online of William Thurston drawing the trefoil and figure-eight knot at the same time, one with each hand. The skill cap is high.



Definition 23.3. Two knots K_1 and K_2 are *equivalent* if there exists a homeomorphism $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that $f(K_1) = K_2$.

This definition doesn't really get at the intuition people naturally have. When people think about knots, they think about rotating and bending knots and loops. However, here is another consideration.

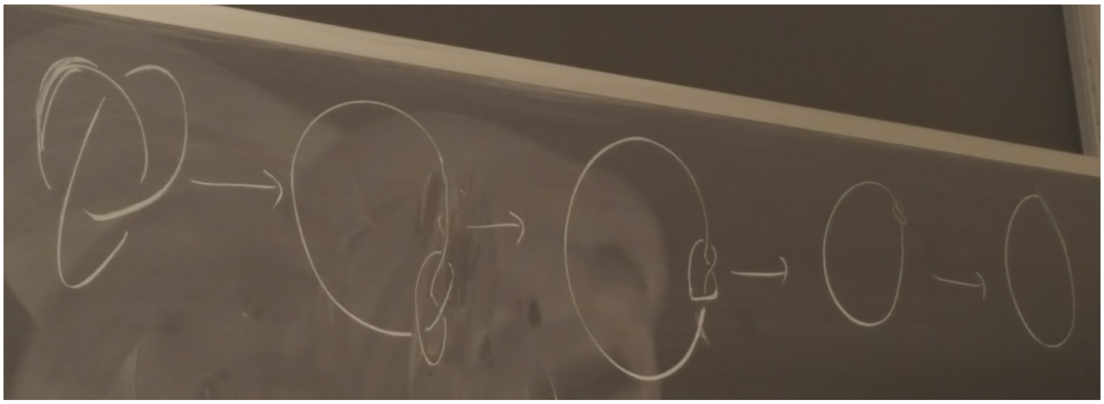
Example 23.2. Take a right-handed trefoil and mirror it across a vertical axis to get a left-handed trefoil.



These trefoils are equivalent by reflection of $\mathbb{R}^3$. Although, one cannot "slide one onto the other" (not obvious).

What is a better notion of equivalence? We need to be careful:

Example 23.3. There exists a continuous family of knots interpolating between the trefoil and the unknot. Keep shrinking the tangled part of the knot.



This works for any knot, not just the trefoil.

Here is a better definition.

Definition 23.4. Two knots K_1 and K_2 are called *isotopic* if there exists a homotopy $F : \mathbb{R}^3 \times [0, 1] \to \mathbb{R}^3$ such that

1. For every $t \in [0, 1]$, $F(x, t) : \mathbb{R}^3 \to \mathbb{R}^3$ is a homeomorphism.
2. $F(x, 0) = \mathrm{id}_{\mathbb{R}^3}$
3. $F(K_1, 1) = K_2$.

Note that isotopic knots are equivalent.

Theorem 23.1. *K_1, K_2 are isotopic iff K_1, K_2 are equivalent via a homeomorphism f that is ambient isotopic to* $\mathrm{id}_{\mathbb{R}^3}$

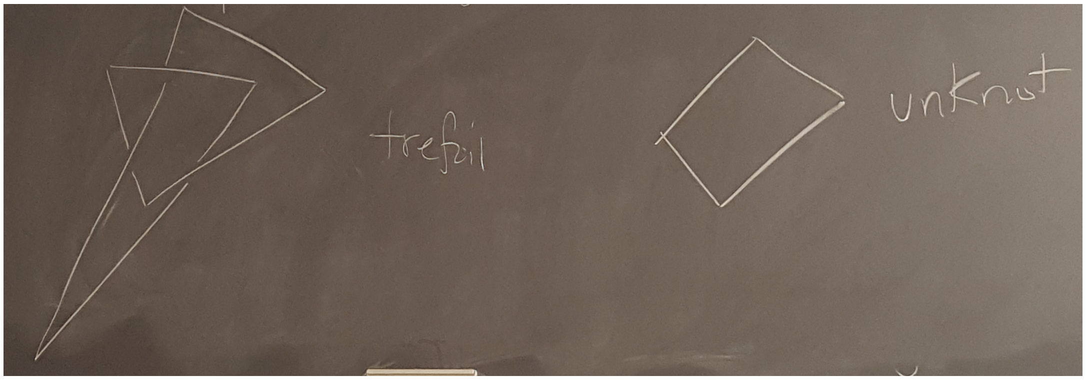
Example 23.4. The right-handed and left-handed trefoil knots are not isotopic.

23.2 Tame knots

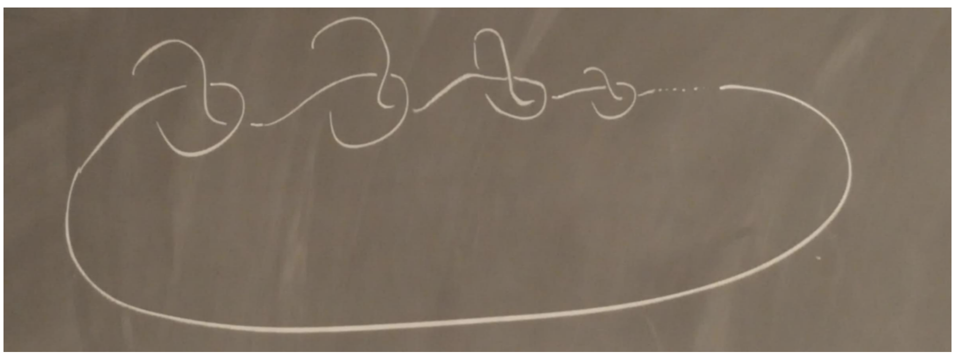
Knots, like topological spaces, can be very weird. We only care about knots that are equivalent to *polygonal knots*.

Definition 23.5. A *polygonal knot* is a knot that is comprised of finitely many line segments. Such knots are called *tame*; otherwise, the knot is called *wild*.

Example 23.5. Here are polygonal knots equivalent to the trefoil and the unknot, respectively.



Example 23.6. Here is an example of a wild knot.



From now on, we will only talk about tame knots. The following theorem says that we can draw tame knots, although we will not prove it.[24]

Theorem 23.2. *Every (tame) knot is isotopic to one whose projection to the (x, y)-plane is nice, i.e. with finitely many double points (two line segments intersecting), no triple (or more) points, and no tangencies.*

If you have studied any differential topology, this is saying that we want intersections to be transverse to each other.

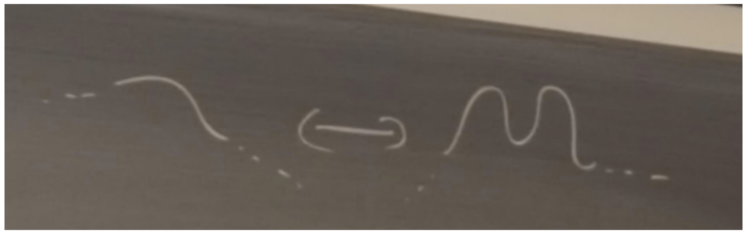
23.3 Reidemeister moves

There are "moves" on a projection that don't change the isotopy class of the corresponding knot.

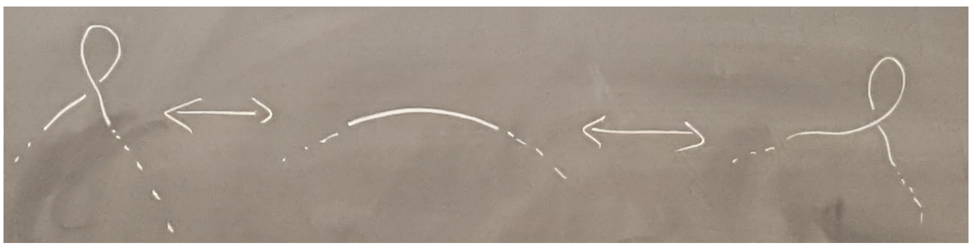
[24] Professor Conway says we won't prove this because it's not fun, and we're having fun right now.

Definition 23.6. The *Reidemeister moves* are the following transformations of a projection of a knot:

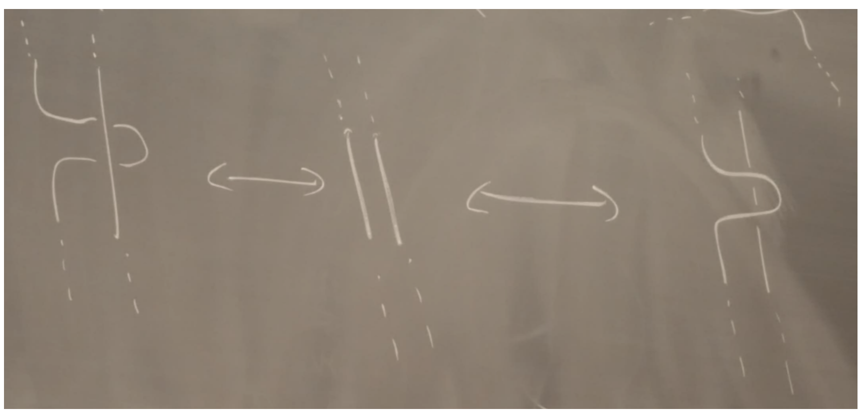
- (R0) We can use any "isotopy in the plane" (homeomorphism of $\mathbb{R}^2$ that is homotopic to $\mathrm{id}_{\mathbb{R}^2}$)



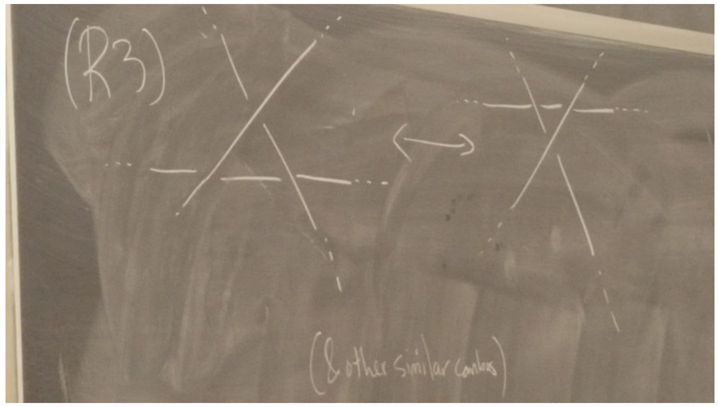
- (R1) We can loop or unloop part of a knot.



- (R2) We can move part of a knot under another part.



- (R3) We can slide part of a knot around if it is below or above a crossing of two other parts of the projection.



Theorem 23.3 (Reidemeister (1927),Alexander-Baird-Briggs (1926)[25])**.** *If two nice projections are of isotopic knots, then the projections are related by a finite sequence of Reidemeister moves (and lots of R(0)).*

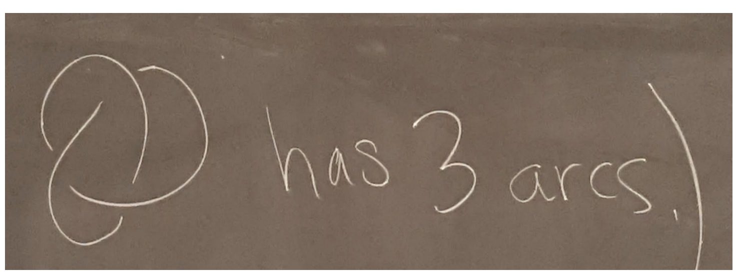
How many moves does it take? We know that it is $\leq 2^{2^{\cdot^{\cdot^{\cdot^{2^n}}}}}$, where n is the number of crossings X in both diagrams and the number of $2s$ is $10^{1000000n}$.[26]

How do we understand knots from their projections, then? We want to define invariants of isotopy classes of knots by defining invariants of nice projections that don't change when doing R1-R3.

23.4 Tricolorability

The idea is to color the arcs of a nice projection in a certain way. We will count the number of ways we can do such a coloring of a given knot. We will explicitly define what an arc is next time.

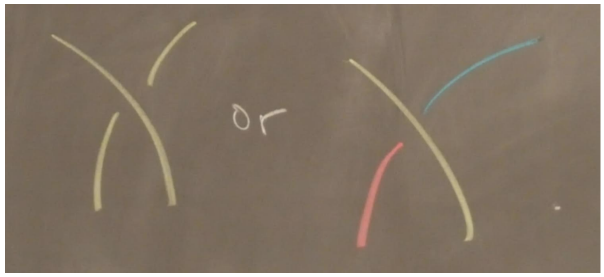
Example 23.7. The trefoil knot has 3 arcs, indicated by how we draw it.



[25] The proofs were done independently.

[26] This is the most current bound, as of 2014.

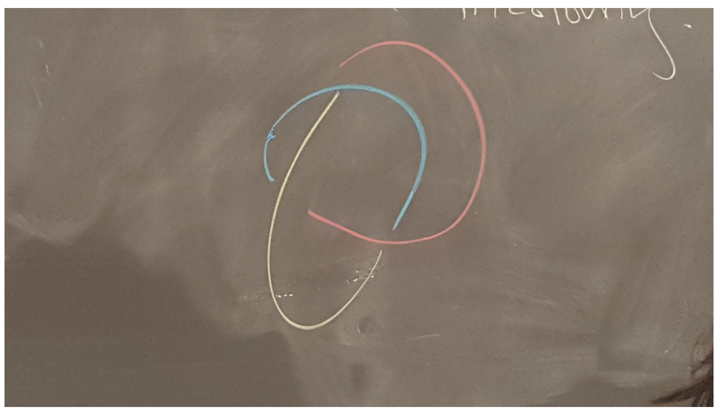
Definition 23.7. A *tricoloring* of a projection is a coloring of the arcs of the projection (red/green/blue or 1/2/3) such that at each crossing, either each color is the same, or all are different.



Definition 23.8. A *trivial coloring* is one that uses one color.

Definition 23.9. A knot is *tricolorable* if there exists a (nice) projection that has a non-trivial tricoloring.

Example 23.8. The trefoil knot is tricolorable.

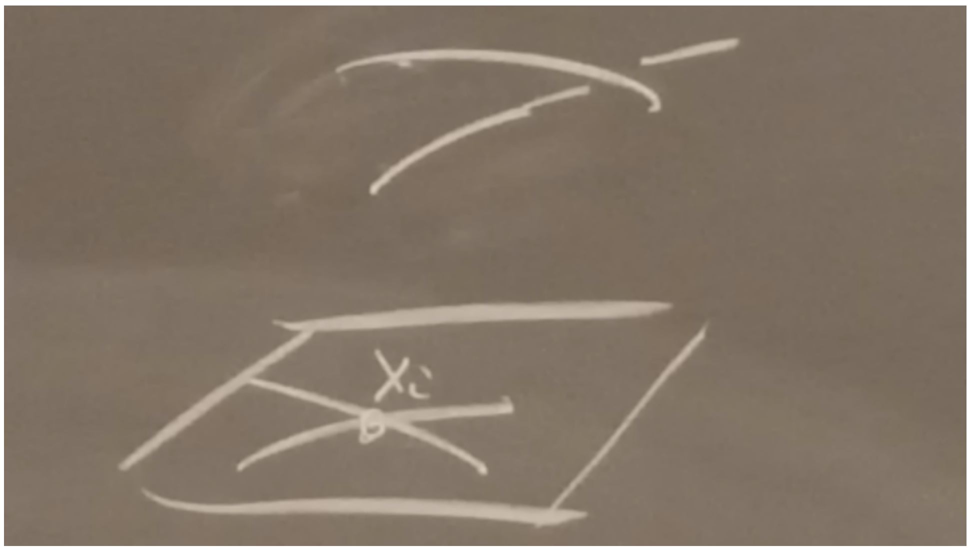


24 Knot Colorings

24.1 Tricolorings

Let's start off by giving a rigorous definition of arc.

Definition 24.1. Given a nice projection, each double point $x_i \in \mathbb{R}^2$ has 2 preimages in K: (x, y, z_1) and (x, y, z_2), where $z_1 < z_2$.



Take a neighborhood $A_i \subseteq K$ for (x, y, z) such that $A_i \cong (0, 1)$ (and is small). Then an *arc* is a connected component of $K \setminus (A_1 \cup \cdots \cup A_n)$.

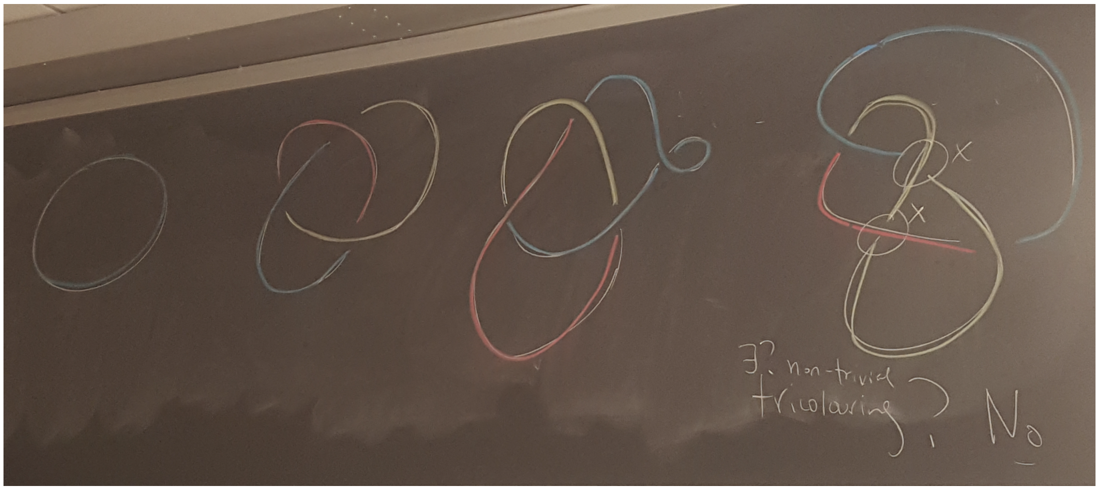
Now that we have this definition, we won't use it ever again.[27]

We defined arcs to talk about tricoloring. Recall that a knot is tricolorable if there exists a nice projection of the knot with a non-trivial tricoloring.

Example 24.1. Let's try to make tricolorings of the unknot, trefoil knot, a modified (by R1) trefoil knot, and a figure-eight knot. If we start with the added loop on the modified trefoil, we must have only 1 color on that crossing (since there are only two arcs involved in the crossing, we cannot have 3 colors). If we start with one arc in the figure-eight knot

[27] I personally lament the sentiment behind this.

and color it blue, we can see that there is no nontrivial tricoloring.

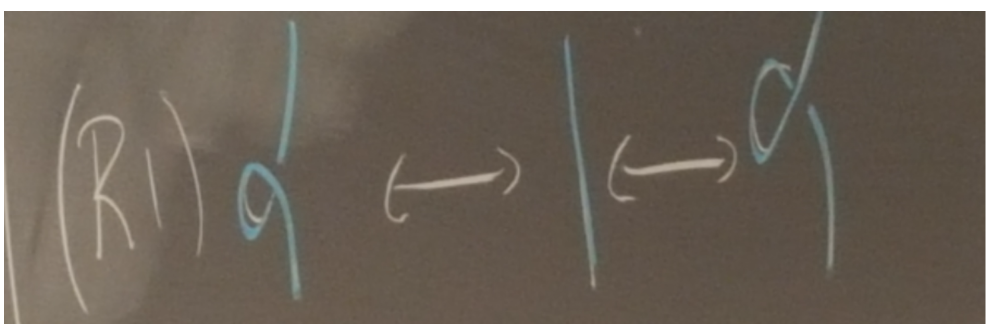


How do we know that we didn't just choose a bad projection of the figure-eight knot? What if another projection is tricolorable?

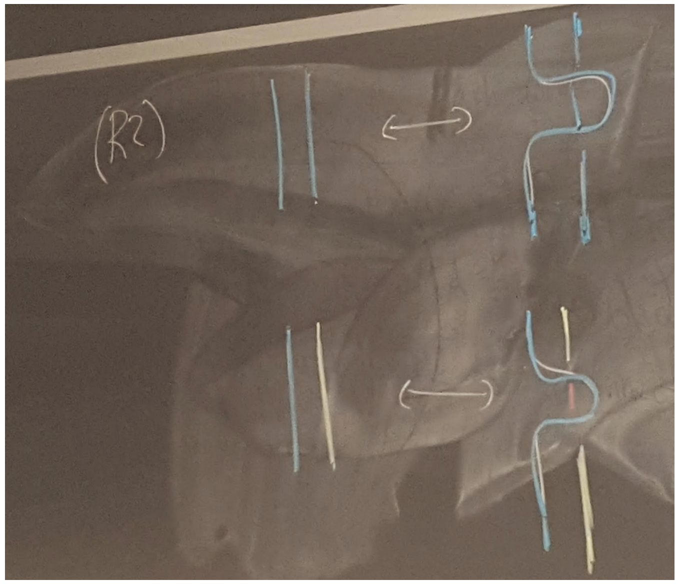
Proposition 24.1. *If a knot is tricolorable, then any nice projection of it (and any knot isotopic to it) has a nontrivial tricoloring.*

Proof. We need to check that the existence of nontrivial tricolorings is independent of projection. So we show it doesn't change under Reidemeister moves.

1. (R1): If we have a self loop crossing, it can only have 1 color because there are only two arcs involved. So

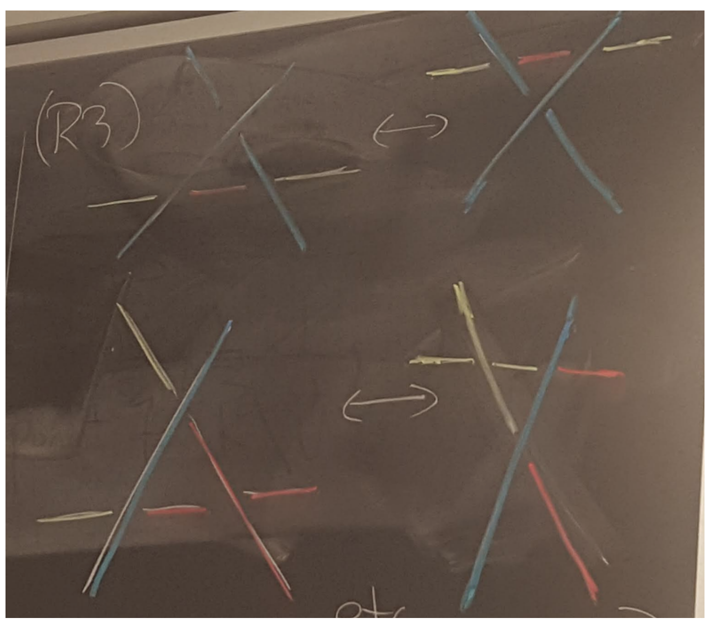


2. (R2): There are two main cases; we leave the rest as an exercise.



   In each case, there is only one choice for the tricoloring of the modified picture; this means that there is a bijection between the tricolorings.

3. (R3) The ends have to be the same color before and after the Reidemeister move so we can "patch in" this section into the knot. This gives us one possible coloring in each case given a coloring before doing the Reidemeister move.



□

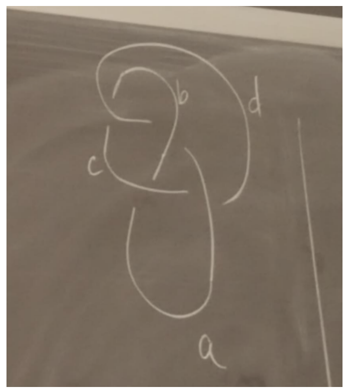
This proof actually shows the following.

Corollary 24.1. *The number of tricolorings of a projection of a knot K is independent of the projection.*

We also get the following corollary.

Corollary 24.2. *The trefoils are not isotopic to the unknot or to the figure-eight knot.*

Proof. The trefoil is tricolorable. The unknot has a projection with no nontrivial colorings, so the proposition implies that the unknot is not tricolorable. The figure-eight knot has a projection



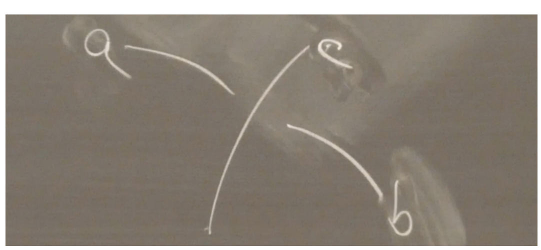
with $a, b, c, d \in \{1, 2, 3\}$. If $a = b$, then one of the crossings makes $a = c$. Then another crossing gives $a = d$, so we get the trivial coloring. If $a \neq b$, say $a = 1$ and $b = 2$, then the first crossing gives us $c = 3$. Another crossing then gives us $d = 2$. But a third crossing contains b, c, and d, so we have a crossing with 2, 3, and 2, which is impossible. Since this projection of the figure-eight knot has no nontrivial tricoloring, the proposition implies that the figure-eight knot is not tricolorable. $\square$

This result leaves us with a question: is the unknot isotopic to the figure-eight knot?

24.2 n-colorings

Let's generalize the idea of 3-colorings.

Definition 24.2. An *n-coloring* of a nice projection is a choice of color $1, \ldots, n$ for each arc such that at each crossing



we have $2c \equiv a + b \pmod{n}$. A *trivial n-coloring* is a one with only one color. A knot is *n-colorable* if there exists a nontrivial n-coloring of a nice projection of a knot isotopic to K.

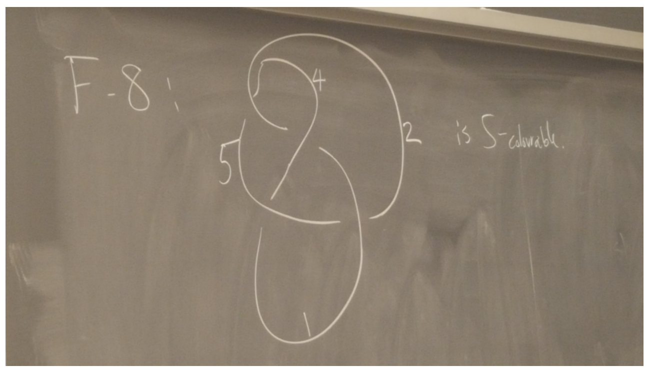
For $n = 3$, $2c - b - a \equiv 0 \pmod{3}$ iff $a = b = c$ or $\{a, b, c\} = \{1, 2, 3\}$. Check this yourself.

Theorem 24.1. *K is n-colorable iff any nice projection of K has a nontrivial n-coloring. The number of n-colorings of a nice projection of K is independent of the projection.*

Proof. The proof is the same as the 3-coloring case. Check (R1)-(R3). □

Corollary 24.3. *The figure-eight knot is not isotopic to the unknot.*

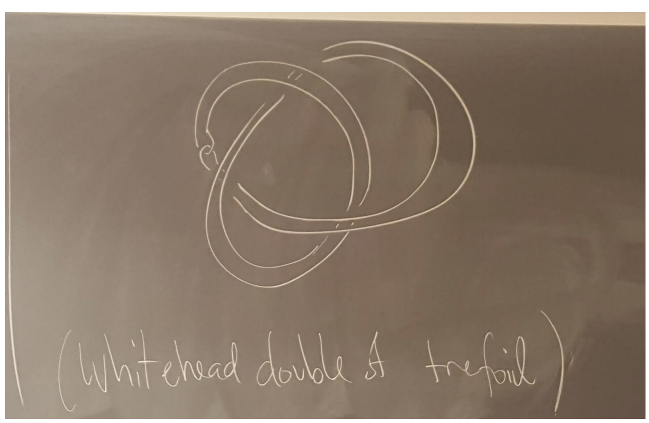
Proof. The unknot is not 5-colorable. The figure-eight knot, however, is 5-colorable. Check that the following 5-coloring works.



Note that we don't have to use all 5 of the colors to get a nontrivial 5-coloring. □

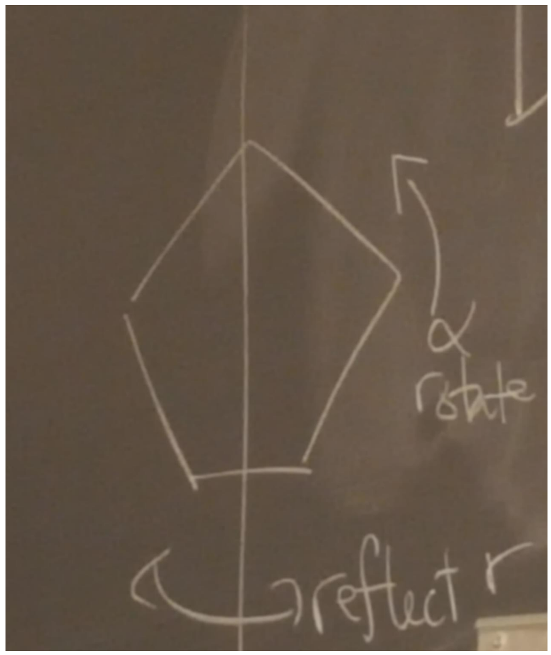
Remark 24.1. There exist nontrivial knots that are not n-colorable for any n.

Example 24.2. Here is a nontrivial knot that is not n-colorable for any n. It is called the *Whitehead double* of the trefoil.



Our goal for the next few lectures is to relate n-colorings to algebraic topology. We will show that the set of n-colorings is almost in bijection with the set of homomorphisms $\pi_1(\mathbb{R}^3 \setminus K) \to D_{2n}$, where D_{2n} is the dihedral group of symmetries of the n-gon.

$$D_{2n} = \left\langle r, \alpha \mid r^2 = 1, \alpha^n - 1, \alpha r = r\alpha^{-1} \right\rangle.$$



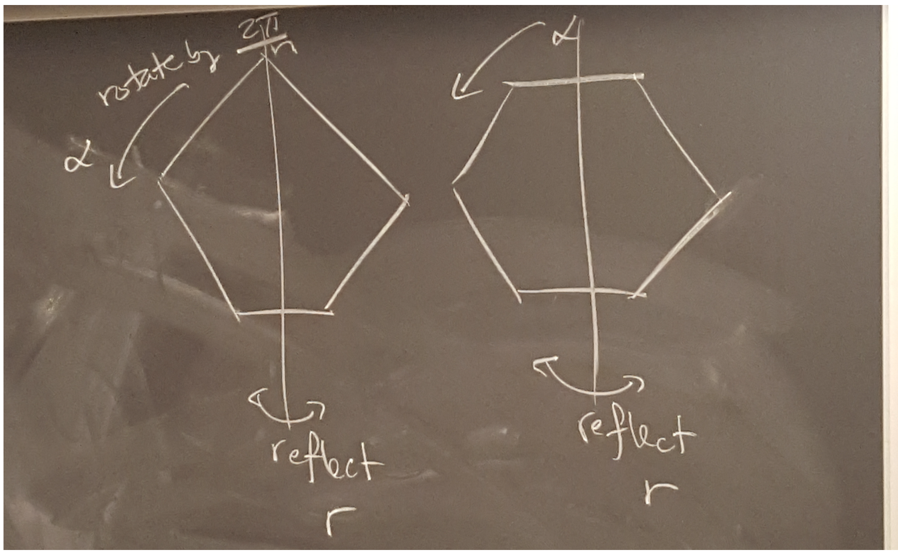
We will calculate $\pi_1(\mathbb{R}^3 \setminus K)$ from a projection. It will have 1 generator x_i for every arc. We will look at homomorphisms $\phi : \pi_1(\mathbb{R}^3 \setminus K) \to D_{2n}$ sending $x_i \mapsto r\alpha^{c_i}$, where c_i is the color for arc i.

25 Knot Colorings in Algebraic Topology

25.1 Knot groups

Our goal is to relate n-colorings to algebraic topology. We will show that n-colorings of K correspond (almost) to homomorphisms $\pi_1(\mathbb{R}^3 \backslash K) \to D_{2n}$; we will actually be overcounting by n. D_{2n} is the group of symmetries of a regular n-gon,[28]

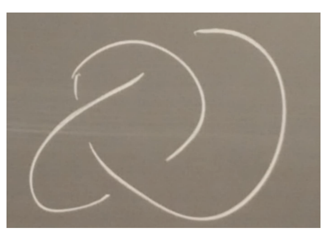
$$D_{2n} = \left\langle r, \alpha \mid \alpha r = r\alpha^{-1}, r^2 = 1, \alpha^n = 1 \right\rangle.$$



We will work out $\pi_1(\mathbb{R}^3 \setminus K)$ to be the following.

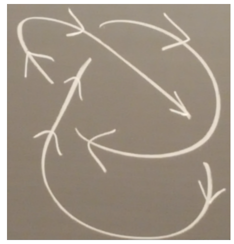
Definition 25.1. The *knot group* of K is constructed by:

1. Take a nice projection of K.



[28]Some people call this group D_n. You should always be clear with your notation when discussing this group.

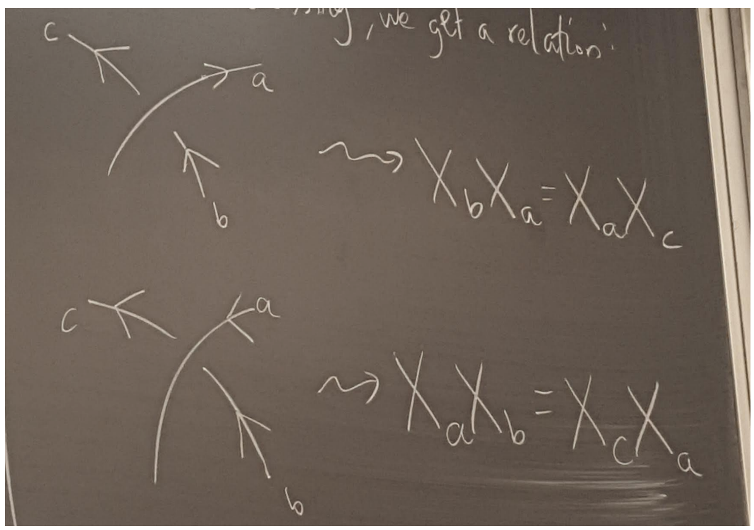
2. Put a direction on the knot.



3. Number the arcs $1, \ldots, n$.

4. $\pi_1(\mathbb{R}^3 \setminus K)$ is generated by $x_1, \ldots, x_n$ (one for each arc).

5. For each crossing, we get a relation:



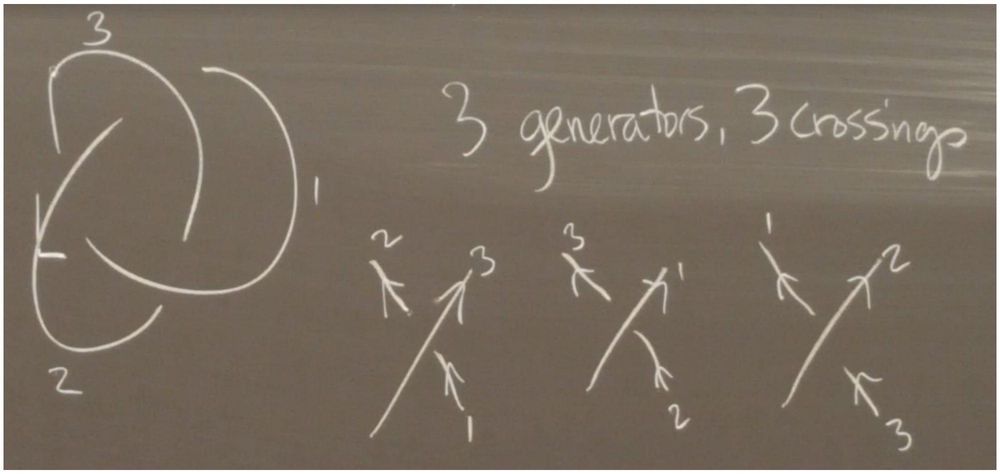
This says that x_c is a conjugate of x_b.

We can also define this for oriented links, but we will not prove that here.

Example 25.1. Let K be the unknot. K has one arc and no crossings, so

$$\pi_1(\mathbb{R}^3 \setminus K) \cong \langle x_1 \rangle \cong \mathbb{Z}.$$

Example 25.2. Let K be the trefoil knot. This has 3 arcs and 3 crossings.



$$\pi_1(\mathbb{R}^3 \setminus K) \cong \langle x_1, x_2, x_3 \mid x_1x_3 = x_3x_2, x_2x_1 = x_1x_3, x_3x_2 = x_2x_1 \rangle$$

Note that $x_3 = x_2x_1x_2^{-1}$, so x_3 is a redundant generator.

$$\cong \left\langle x_1, x_2 \mid x_1x_2x_1x_2^{-1} = x_2x_1x_2^{-1}x_2, x_2x_1 = x_1x_2x_1x_2^{-1} \right\rangle$$

Write $a = x_1$, $b = x_2$, and simplify.

$$\cong \langle a, b \mid aba = bab, bab = aba \rangle$$

These relations are redundant.

$$\cong \langle a, b \mid aba = bab \rangle .$$

In general, it is hard to tell apart groups like this by their generators. Before, we used Abelianization to tell apart fundamental groups. However, that approach doesn't work here.

Proposition 25.1. *Let K be a knot. Then*

$$\mathrm{Ab}(\pi_1(\mathbb{R}^3 \setminus K)) \cong \mathbb{Z}.$$

Proof. Every relation is $x_ax_b = x_bx_c$. This reorders to $x_ax_b = x_cx_b$, which gives us that $x_a = x_c$. Check that all generators are identified this way. □

Regardless, we are looking at the right object.

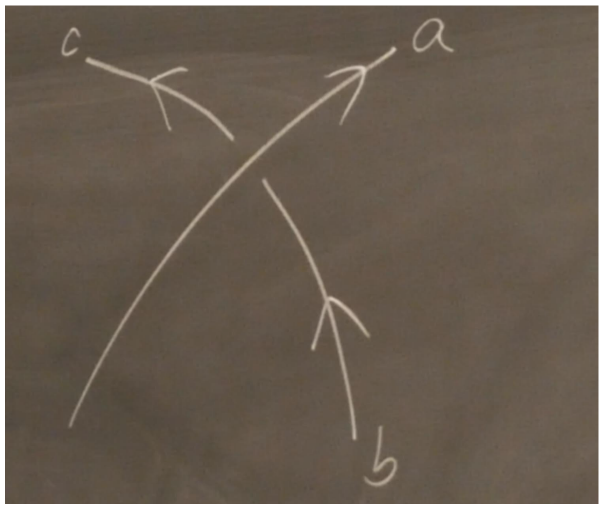
Theorem 25.1 (Gordon-Luecke, 1989)**.** *$\pi_1(\mathbb{R}^3 \setminus K_1) \cong \pi_1(\mathbb{R}^3 \setminus K_2)$ iff K_1 and K_2 are equivalent.*

So this fundamental group determines the knot up to isotopy or mirroring.

25.2 Correspondence between knot colorings and fundamental group homomorphisms

Theorem 25.2. *Let K be a knot. Then there is a correspondence between n-colorings of K and homomorphisms $\pi_1(\mathbb{R}^3 \setminus K) \to D_{2n}$ (except for n homomorphisms).*

Proof. Given an n-coloring that sends arc i to color $\ell_i \in \{1, \ldots, n\}$, construct the homomorphism $\pi_1(\mathbb{R}^3 \setminus K) \to D_{2n}$, via $x_i \mapsto r\alpha^{\ell_i}$. We need to check the relations. At a crossing



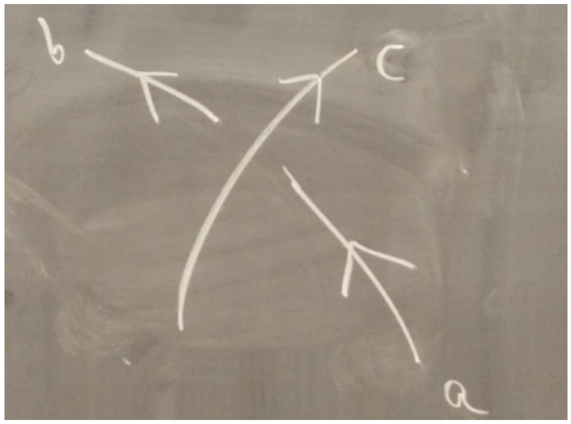
we know that $x_b x_a = x_a x_c$. We get

$$x_b x_a \mapsto r\alpha^{\ell_b} r\alpha^{\ell_a} = rr\alpha^{-\ell_b}\alpha^{\ell_a} = r^2\alpha^{\ell_a - \ell_b} = \alpha^{\ell_a - \ell_b},$$

$$x_a x_c \mapsto r\alpha^{\ell_a} r\alpha^{\ell_c} = rr\alpha^{-\ell_a}\alpha^{\ell_c} = r^2\alpha^{\ell_c - \ell_a} = \alpha^{\ell_c - \ell_a}.$$

We want $\alpha^{\ell_a - \ell_b} = \alpha^{\ell_c - \ell_a}$; i.e. we need $\ell_a - \ell_b \equiv \ell_c - \ell_a \pmod{n}$. This is equivalent to $2\ell_a \equiv \ell_b + \ell_c \pmod{n}$, which is the requirement for an n-coloring.

The argument is similar for the other crossing type. So we have a homomorphism. We can also go backward (homomorphism to coloring), but only if for all i, $\phi(x_i) = r\alpha^{\ell_i}$ for some ℓ_i; ℓ_i will be the color of arc i.If $\phi(x_a) = \alpha^{\ell_i}$ for some a, we need $\phi(x_a x_c) = \phi(x_c x_b)$.



Count how many reflections we have on the left hand side and on the right.

LHS # refls.	$\phi(x_c)$
1	$r\alpha^{\ell_c}$
0	α^{ℓ_c}

RHS # refls.	$\phi(x_c)$	$\phi(x_b)$
0	$r\alpha^{\ell_c}$	$r\alpha^{\ell_b}$
1	$r\alpha^{\ell_c}$	α^{ℓ_b}
1	α^{ℓ_c}	$r\alpha^{\ell_b}$
0	α^{ℓ_c}	α^{ℓ_b}

In either case, $\phi(x_b) = \alpha^{\ell_b}$ for some ℓ_b. Follow our knot around, doing the same analysis at every crossing. Then $\phi(x_i) = \alpha^{\ell_i}$ for all i.

Now $\phi(x_a x_b) = \phi(x_c x_b)$ iff $\alpha^{\ell_a}\alpha^{\ell_c} = \alpha^{\ell_c}\alpha^{\ell_b}$. This is the condition that $\ell_a \equiv \ell_b \pmod{n}$. Check that this makes $\phi(x_i) = \phi(x_j)$ for al i, j. So ignore these homomorphisms (there are n of them). Hence,

$$|\{n\text{-colorings of } K\}| = |\{\text{homomorphisms } \pi_1(\mathbb{R}^3 \setminus K) \to D_{2n}\}| - n. \qquad \square$$

Here is an application of this result.

Example 25.3. Let K be a knot siting on a torus that gives the element $3, 4 \in \pi_1(T^2)$; this goes 3 times around the torus and 4 times through the center hole. We can show that $\pi_1(\mathbb{R}^3 \setminus K) \cong \langle a, b \mid a^3 = b^4 \rangle$.

K is not n colorable for any n. If $\phi : \pi_1(\mathbb{R}^3 \setminus K) \to D_{2n}$ is a homomorphism such that $\phi(a) = r\alpha^i$ and $\phi(b) = r\alpha^j$, we need that $\phi(a)^3 = \phi(b)^4$. But $\phi(a)^3 = r\alpha^i r\alpha^i r\alpha^i = r\alpha^i$ and $r\alpha^j r\alpha^j r\alpha^j r\alpha^j = 1$. So we have a contradiction.