1 Properties of Subharmonic Functions

1.1 Local conditions equivalent to subharmonicity

Last time, we introduced the notion of a subharmonic function.

**Theorem 1.1.** Let $u : \Omega \to (-\infty, \infty)$ be upper semicontinuous. The following are equivalent:

1. $u$ is subharmonic.

2. If $\{|x - a| \leq R\} \subseteq \Omega$, then

   $$u(a) \leq \frac{1}{2\pi R} \int_{|y| = R} P_R(x - a, y) u(a + y) \, ds(y).$$

3. (local sub-mean value inequality): For every $a \in \Omega$,

   $$u(a) \leq \frac{1}{2\pi R} \int_{|y| = R} u(a + y) \, ds(y)$$

   for all small $R > 0$.

4. For every $a \in \Omega$,

   $$u(a) \leq \frac{1}{\pi R^2} \iint_{|y| \leq R} u(a + y) \, dy$$

   for all small $R > 0$, where $dy$ is Lebesgue measure in $\mathbb{R}^2$.

5. If $\{|x - a| \leq R\} \subseteq \Omega$, then

   $$u(a) \leq \frac{1}{\pi R^2} \iint_{|y| \leq R} u(a + y) \, dy$$

**Remark 1.1.** It follows from properties 3 and 4 that subharmonicity is a local property.
Remark 1.2. The integrals in the theorem are Lebesgue integrals of upper semicontinuous functions. If \( u : \Omega \to [-\infty, \infty) \) is upper semicontinuous and \( K \subseteq \Omega \) is compact, then

\[
\int_K u(x) \, dx = \inf_{\varphi \in C(K)} \int \varphi \, dx \in [-\infty, \infty).
\]

Proof. (1) \( \implies \) (2): Let \( f \in C(|x-a|=R) \), and let \( v \in C(|x-a| \leq R) \) be harmonic in \(|x-a| < R\) so that \( v = f \) along \(|x-a| = R\). If \( u \leq f \) on \(|x-a| = R\), then \( u \leq v \) in \(|x-a| \leq R\). So

\[
u(x) \leq \frac{1}{2\pi R} \int_{|y|=R} P_R(x-a,y) f(a+y) \, ds(y)
\]

for \(|x-a| < R\). Pick a sequence \( f_k \in C(|x-a|=R) \) such that \( f_k \downarrow u \). Apply this inequality to every function in the sequence, and let \( k \to \infty \) by monotone convergence to get the desired inequality.

(2) \( \implies \) (3): Take \( x = a \).

(2) \( \implies \) (5): If \( \{|x-a| \leq R\} \), then

\[
u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{it}) \, dt
\]

with \( 0 < r \leq R \). Multiply by \( 2r \) and integrate over \([0, R]\). This gives us the area integral, expressed in polar coordinates.

(5) \( \implies \) (4): This is a special case.

(3) \( \implies \) (1): Let \( K \subseteq \Omega \) be compact and \( h \in C(K) \cap H(K^c) \) such that \( u \leq h \) on \( \partial K \). We want to show that \( u \leq h \) on \( K \). The function \( u-h \) is upper semicontinuous on \( K \) and satisfies the local sub-mean value inequality in \( K \). We can prove the maximum principle for \( u-h \) on \( K \) with the same proof as for harmonic functions: If \( M = \max_K (u-h) \), then the set \( \{x \in K : u(x) - h(x) = M\} \) is closed (as \( u-h \) is upper semicontinuous on \( K \)). We get that \( \max_K u-h = \max_{\partial K} h \leq 0 \). So \( u \leq h \) on \( K \).

(4) \( \implies \) (1): The argument is similar to the proof of (3) \( \implies \) (1), using the local sub-mean value inequality with respect to small discs rather than circles.

\[\square\]

1.2 Mean value property and maximum principle

In the proof of the theorem, we also proved the following property.

**Theorem 1.2** (mean value property for subharmonic functions). Let \( \Omega \subseteq \mathbb{R}^2 \) be open and bounded, and let \( u \) be upper semicontinuous on \( \overline{\Omega} \) and subharmonic in \( \Omega \). Then

\[
\max_{\partial \Omega} u = \max_{\overline{\Omega}} u.
\]

We also have the following version of the maximum principle.
Theorem 1.3 (maximum principle for subharmonic functions). Let $\Omega \subseteq \mathbb{R}^2$ be open and connected, and let $u$ be subharmonic $\Omega$. If $u$ contains a global maximum on $\Omega$, then it is constant.

Proof. Let $M = \max_{\Omega} u$, and notice that the sets $\{u < M\}$, $\{u = M\}$ are open. It is important to note that the maximum needs to be global. In this sense, subharmonic functions are much less rigid than their harmonic counterparts.

Example 1.1. Here is an example where $u$ attains a local maximum without being constant in $\Omega$. Take $u(z) = \max(0, \text{Re}(z))$.

1.3 Relationship to holomorphic functions

Proposition 1.1. Let $\Omega \subseteq \mathbb{C}$ be open, and let $f \in \text{Hol}(\Omega)$. Then $u = \log |f| : \Omega \to [-\infty, \infty)$ is subharmonic in $\Omega$.

Proof. We saw before that $u$ is upper semicontinuous, and we shall check that for all $a \in \Omega$, $u(a) \leq \frac{1}{2\pi R} \int_{|y|=R} u(a + y) \, ds(y)$ for all small $R > 0$. If $f(a) = 0$, then the inequality holds. If $f \neq 0$, then in a small simply connected neighborhood of $a$, we can write $u = \text{Re}(\log(f))$. Then $u$ is harmonic near $a$ and the inequality holds with an equality for all $R > 0$.

Next time, we will prove the following result.

Proposition 1.2. Let $f \in C(|z| \leq R) \cap \text{Hol}(|z| < R)$. Assume that there exists a Lebesgue measurable $E \subseteq \{|z| = R\}$ of positive measure such that $f|_E = 0$. Then $f \equiv 0$ in $|z| < R$. 