1 The Phragmén-Lindelöf Principle

1.1 The Phragmén-Lindelöf Principle for subharmonic functions

To prove the Phragmén-Lindelöf principle, let’s introduce some notation.

**Definition 1.1.** Let \( \Omega \subseteq \mathbb{R} \) be open and unbounded. We say that \( \varphi : \overline{\Omega} \to \mathbb{R} \) is a Phragmén-Lindelöf function for \( \Omega \) if

1. \( \varphi(x) > 0 \) for large \( |x| \).
2. If \( u \) is upper semicontinuous on \( \overline{\Omega} \), subharmonic in \( \Omega \), \( u \leq M \) on \( \partial \Omega \), and \( u(x) \leq \varphi(x) \) for large \( x \in \overline{\Omega} \), then \( u \leq M \) on \( \overline{\Omega} \).

**Remark 1.1.** Let \( \varphi \) be a PL function for \( \Omega \). Let \( f \in \text{Hol}(\Omega) \cap C(\overline{\Omega}) \) be such that \( |f| \leq M \) on \( \partial \Omega \) and \( |f(z)| \leq e^{\varphi(z)} \) for large \( z \in \overline{\Omega} \). Then \( |f| \leq M \) on \( \overline{\Omega} \).

Given \( \Omega \), how do we construct PL functions for \( \Omega \)?

**Theorem 1.1** (Phragmén-Lindelöf principle). Let \( \Omega \subseteq \mathbb{R}^2 \) be open and unbounded. Let \( \psi : \overline{\Omega} \to [0, \infty) \) be such that

1. \( \psi \) is lower semicontinuous on \( \Omega \) (\( -\psi \) is upper semicontinuous),
2. \( \psi \) is super harmonic in \( \Omega \) (\( -\psi \) is subharmonic),
3. \( \psi(x) \to +\infty \) as \( |x| \to \infty \) for \( x \in \overline{\Omega} \).

Let \( \varphi > 0 \) be such that \( \varphi(x) = o(\psi(x)) \) when \( |x| \to \infty \) for \( x \in \overline{\Omega} \). Then \( \varphi \) is a PL function for \( \Omega \).

Here is the original argument by Phragmén and Lindelöf.

\textsuperscript{1}Lindelöf was the teacher of Ahlfors.
Proof. Let \( u \) be upper semicontinuous on \( \overline{\Omega} \), subharmonic in \( \Omega \), \( u \leq M \) on \( \partial \Omega \), and \( u(x) \leq \varphi(x) \) for large \( x \in \overline{\Omega} \). We want to show that \( u \leq M \) on \( \overline{\Omega} \). For \( \varepsilon > 0 \), set \( u_\varepsilon = u - \varepsilon \psi \). Then \( u_\varepsilon \) is upper semicontinuous on \( \overline{\Omega} \), subharmonic in \( \Omega \), \( u_\varepsilon \leq M \) on \( \partial \Omega \), and for large \( x \in \overline{\Omega} \),

\[
 u_\varepsilon(x) \leq \varphi(x) - \varepsilon \psi(x) = -\psi(x) \left( \varepsilon - \frac{\varphi(x)}{\psi(x)} \right) |x| \xrightarrow{|x|\to\infty} -\infty.
\]

Let \( a \in \Omega \), and let \( R > |a| \) be such that \( u_\varepsilon(x) \leq M \) for \( |x| = R \) and \( x \in \overline{\Omega} \). If \( \Omega_R = \{ x \in \Omega : |x| < R \} \), then \( \partial \Omega \subseteq \partial \Omega \cup \{ x \in \overline{\Omega} : |x| = R \} \), and \( u_\varepsilon \leq M \) on \( \partial \Omega_R \). Apply the maximum principle to \( u_\varepsilon \) and the bounded domain \( \Omega_R \) to get that \( u_\varepsilon \leq M \) on \( \Omega_R \). So

\[
 u_\varepsilon(a) = u(a) - \varepsilon \psi(a) \leq M.
\]

Letting \( \varepsilon \to 0^+ \), we get that \( u \leq M \) on \( \Omega \). So \( \varphi \) is a PL function for \( \Omega \).

\[ \square \]

### 1.2 Phragmén-Lindelöf for a sector

This important case of the theorem was the original motivation for Phragmén and Lindelöf.

**Theorem 1.2 (PL for a sector).** Let \( \Omega = \{ z \in \mathbb{C} \setminus \{ 0 \} : \alpha < \arg(z) < \beta \} \) for \( 0 < \beta - \alpha < 2\pi \). Then \( \varphi(z) = |z|^k \) is a PL function for \( \Omega \) if \( 0 < k < \pi/(\beta - \alpha) \).

**Proof.** We may assume after a rotation that \( \Omega = \{ z \in \mathbb{C} \setminus \{ 0 \} : |\arg(z)| < \gamma/2 \} \), where \( 0 < \gamma = \beta - \alpha < 2\pi \). Let \( k < k_1 < \pi/\gamma \), and consider \( \psi(z) = \text{Re}(z^{k_1}) = \text{Re}(e^{k_1 \log(z)}) \), using the principal branch of log. This is \( \psi(z) = |z|^{k_1} \cos(k_1 \arg(z)) \) for \( z \in \overline{\Omega} \) with \( z \neq 0 \). Then \( \psi \) is harmonic in \( \Omega \), continuous in \( \overline{\Omega} \), and \( |\psi(z)| \sim |z|^{k_1} \) since \( |k_1 \arg(z)| \leq k_1 \gamma/2 < \pi/2 \). In particular, \( \phi = o(\psi) \) at \( \infty \). Therefore, \( \varphi \) is a PL function for \( \Omega \).

**Corollary 1.1 (classical PL principle).** Let \( \Omega = \{ z \in \mathbb{C} \setminus \{ 0 \} : \alpha < \arg(z) < \beta \} \), where \( 0 < \beta - \alpha < 2\pi \). Let \( f \in \text{Hol}(\Omega) \cap C(\overline{\Omega}) \), where \( |f| \leq M \) on \( \partial \Omega \). Assume that \( |f(z)| \leq C_1 e^{C_2 |z|^k} \) as \( |z| \to \infty \) for \( z \in \overline{\Omega} \), where \( 0 < k < \pi/(\beta - \alpha) \). Then \( |f| \leq M \) on \( \overline{\Omega} \).

Here is an example from the spring 2015 analysis qualifying exam.

**Example 1.1.** Let \( f \in \text{Hol}(\mathbb{C}) \) be such that \( |f(z)| \leq e^{|z|} \) and \( \sup_{x \in \mathbb{R}} (|f(x)|^2 + |f(ix)|^2) < \infty \). Show that \( f \) is constant.

Apply the classical Phragmén-Lindelöf principle 4 times, once to each quadrant. Then \( f \) is bounded, so \( f \) is constant by Liouville’s theorem.

### 1.3 Phragmén-Lindelöf for general domains

Let \( \Omega, \Omega' \subseteq \mathbb{C} \) be open and unbounded, and let \( G : \Omega \to \Omega' \) is an analytic isomorphism such that \( G \) extends to a homeomorphism \( \overline{\Omega} \to \overline{\Omega'} \). Then \( |G(z)| \) is large iff \( |z| \) is large. Then if \( \varphi \) is a PL function for \( \Omega' \), \( \varphi \circ G \) is a PL function for \( \Omega \). (To check this, use that if \( u \in \text{SH}(\Omega') \), then \( u \circ G \in \text{SH}(\Omega) \).)
Proposition 1.1. Let $\Omega = \{z \in \mathbb{C} : \text{Im}(z) > 0, \alpha < \text{Re}(z) < \beta\}$. Then $\varphi(z) = e^{k\text{Im}(z)}$ is a PL function for $\Omega$ for any $0 < k < \pi/(\beta - \alpha)$.

We will prove this next time. The idea is that we find a conformal map from the half-strip to a sector with a disc removed. The map is $f(z) = e^{-icz}$ for some $0 < c < 2\pi/(\beta - \alpha)$. 