1 Hartogs’ Theorem

1.1 Lemmas containing the argument

The goal is to prove the following theorem.

Theorem 1.1 (Hartogs). Let $\Omega \subseteq \mathbb{C}^n$ be open, and let $u : \Omega \to \mathbb{C}$ be separately holomorphic. Then $u \in \text{Hol}(\Omega)$.

We will break up the proof into a few lemmas.

Lemma 1.1. Let $\Omega \subseteq \mathbb{C}^n$ be open, and let $u$ be separately holomorphic in $\Omega$. If $u$ is locally bounded in $\Omega$, then $u \in C(\Omega)$ (so $u \in \text{Hol}(\Omega)$).

Proof. Let $D$ be a polydisc with $\overline{D} \subseteq \Omega$. Write $D = D_1 \times D'$, where $D_1$ is a disc in $\mathbb{C}$ and $D'$ is a polydisc in $\mathbb{C}^{n-1}$. The function $z_1 \mapsto u(z_1, z') \in \text{Hol}(D_1)$. By Cauchy’s integral formula, $\partial_{z_j} u(z_1, z')$ is bounded when $z_1 \in D_1' \subseteq D_1$ (compactly contained) and $z' \in D'$. It follows that $\partial_{z_j} u$ is bounded on a relatively compact polydisc $\subseteq D$; in other words, $\partial_{z_j} u$ are locally bounded in $\Omega$. Also, $\partial_{z_j} = 0$ for all $j$.

It follows that $u$ is continuous. If $a \in \Omega$ and $h \in \mathbb{C}^n \cong \mathbb{R}^{2n}$,

$$ u(a + h) - u(a) = \sum_{j=1}^{2n} u(a + v_j) - u(a + v_{j-1}), \quad v_j = (h_j, \ldots, h_j, 0, \ldots, 0). $$

Now use the mean value theorem. □

Induction on $n$: Now assume that Hartogs’ theorem is already known for functions of $< n$ complex variables.

Lemma 1.2. Let $u : \Omega \to \mathbb{C}$ be separately holomorphic, and let $D = \prod_{j=1}^n D_j$ be a closed polydisc $\subseteq \Omega$ with $D^0 \neq \emptyset$. Then there exist discs $D'_j \subseteq D_j$ for $1 \leq j \leq n - 1$ with nonempty interior such that if $D'_n = D_n$, then $u$ is bounded on $D' = \prod_{j=1}^n D'_j$. 

Proof. Let \( E_M = \{ z' \in \prod_{j=1}^{n-1} D_j : |u(z', z_n)| \leq M \forall z_n \in D_n \} \). \( E_M \) is closed: by the inductive hypothesis, \( z' \mapsto u(z', z_n) \) is holomorphic in a neighborhood of \( \prod_{j=1}^{n-1} D_j \) for each \( z_n \) and thus continuous; so

\[
E_M = \bigcap_{z_n \in D_n} \left\{ z' \in \prod_{j=1}^{n-1} D_j : |u(z', z_n)| \leq M \right\}
\]

is an intersection of closed sets. Also, \( \bigcup_{M=1}^{\infty} E_M = \prod_{j=1}^{n-1} D_j : z_n \mapsto u(z', z_n) \) is holomorphic near \( D_n \) for all \( z' \in \prod_{j=1}^{n-1} D_j \) and is thus bounded on \( D_n : |u(z', z_n)| \leq M \) for \( z_n \in D_n \).

\( \prod_{j=1}^{n-1} D_j \) is a complete metric space; so by Baire’s theorem, so \( E_M \) has nonempty interior for some \( M \). So \( E_M \) contains a polydisc \( D' = \prod_{j=1}^{n-1} D'_j \) with nonempty interior such that if \( D'_n = D_n, u \) is bounded in \( D' = \prod_{j=1}^{n} D'_j \subseteq D' \).

**Lemma 1.3.** Let \( D \) be a polydisc \( \{ |z_j - z_j| < R : j = 1, \ldots, n \} \). Let \( u : D \to \mathbb{C} \) be holomorphic in \( z' = (z_1, \ldots, z_{n-1}) \) for every fixed \( z_n \), and assume that \( u \) is holomorphic and bounded in \( D' \) given by \( |z_j - z^0_j| < r \) for all \( 1 \leq j \leq n - 1 \) for some \( r > 0 \) and \( |z_n - z^0_n| < R \). Then \( u \in \text{Hol}(D) \).

**Proof.** We may assume that \( z^0 = 0 \). Take \( 0 < R_1 < R_2 < R \). Taylor expand \( z' \mapsto u(z', z_n) \):

\[
u(z', z_n) = \sum_{\alpha' \in \mathbb{N}^{n-1}} a_{\alpha'}(z_n)(z')^{\alpha'}, \quad |z_j| < R, 1 \leq j \leq n - 1, |z_n| < R.
\]

We have that

\[ a_{\alpha'}(z_n) = \left. \frac{\partial^{\alpha'}(0, z_n)}{(\alpha')!} \right|_{z_n = 0} \]

is holomorphic in \( |z_n| < R \). This series converges normally in \( |z_j| < R \) for \( 1 \leq j \leq n - 1 \). So \( a_{\alpha'}(z_n) R_2^{|\alpha'|} \to 0 \) as \( |\alpha'| \to \infty \) for each \( z_n \). Now we have that \( |u| \leq M \) in \( D' \). By Cauchy’s estimates in \( z' \), we know that \( |a_{\alpha'}(z_n)| \leq \frac{M}{r^{\alpha}} \quad \forall \alpha' \).

Consider the sequence of subharmonic (in \( |z_n| < R \)) functions

\[
\varphi_{\alpha'}(z_n) = \frac{1}{|\alpha'|} \log |a_{\alpha'}(z_n)|, \quad |\alpha'| = \alpha_1 + \cdots + \alpha_{n-1}.
\]

Our bound gives us that \( \varphi_{\alpha'} \) is uniformly bounded above in \( |z_n| < R \). Since \( a_{\alpha'}(z_n) R_2^{|\alpha'|} \to 0 \) as \( |\alpha'| \to \infty \),

\[
\lim_{|\alpha'| \to \infty} \sup_{z_n} \varphi_{\alpha'}(z_n) \leq \log(1/R_2)
\]

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for all \( z_n \). By Hartogs’ lemma on subharmonic functions, if \( |z_n| \leq R_n \), then for any \( \varepsilon > 0 \),
\[
\varphi_{\alpha'}(z_n) \leq \log(1/R_2) + \varepsilon \leq \log(1/R_1)
\]
for large \( |\alpha'| \). In other words, for large \( |\alpha'| \) and \( |z_n| \leq R_2 \),
\[
|a_{\alpha'}(z_n)|r_1|z_1^{\alpha_1}| \leq 1.
\]
The series \( \sum_{\alpha' \in \mathbb{N}^{n-1}} a_{\alpha'}(z_n)(z')^\alpha \) converges absolutely for \( |z_n| < R_2 \) and \( |z_j| < R_1 \) (for all \( 1 \leq j \leq n-1 \)) and hence normally in \( D \). So \( u \in \text{Hol}(D) \) as a limit of holomorphic functions (the partial sums).

1.2 Proof of the theorem from the lemmas

We can now prove Hartogs’ theorem.

Proof. Let \( z^0 \in \Omega \), and take a closed polydisc \( \{|z_j - z^0_j| < 2R, 1 \leq j \leq n \} \). Apply the second lemma to the closed polydisc with \( |z_j - z^0_j| \leq R \) for \( 1 \leq j \leq n-1 \) and \( |z_n - z^0_n| \leq 2R \).

Then we get a polydisc of the form \( |z_j - \zeta^0_j| < r \) for \( 1 \leq j \leq n-1 \) and \( |z_n - z^0_n| < R \) with \( \{|z_j - \zeta^0_j| < r \} \subseteq \{|z_j - z^0_j| < R, 1 \leq j \leq n-1 \} \) such that \( u \) is holomorphic and bounded there. In particular, \( |z_j - z^0_j| \). In particular, \( |\zeta^0_j - z^0_j| < R \).

Consider the polydisc \( D \) given by \( |z_j - \zeta^0_j| < R \) for \( 1 \leq j \leq n-1 \) and \( |z_n - z^0_n| < R \) (closure in \( \Omega \)): in the polydisc, \( u \) is holomorphic in \( z' \) if \( z_n \) is fixed, and \( u \) is holomorphic and bounded in the polydisc \( |z_j - \zeta^0_j| < r \) for \( j = 1, \ldots, n \) and \( |z_n - z^0_n| < R \). By the third lemma, \( u \) is holomorphic in \( D \), which is a neighborhood of \( z_0 \).