Math 255A' Lecture 14 Notes

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1 Adjoints

1.1 Adjoints of linear maps

If $T : X \to Y$ is a linear map, then $f \mapsto f \circ T$ is a lienar operator on linear functionals. If T is bounded, then $f \circ T$ is continuous, so this restricts to a linear map $T^* : Y^* \to X^*$.

Definition 1.1. T^* is called the **adjoint** of T.

Proposition 1.1. *If T is bounded, then $\|T^*\| \leq \|T\|$.*

Proof.

$$\begin{aligned} \|T^* f\| &= \sup\{|T^* f(x)| : \|x\|_X \leq 1\}\} \\ &= \sup\{|f(Tx)| : \|x\|_X \leq 1\}\} \\ &\leq \|f\|\|T\|. \end{aligned}$$

□

Proposition 1.2. *Let X, Y be normed spaces, and let $T : X \to Y$ be linear. The following are equivalent:*

1. *T is bounded.*
2. *$f \circ T \in X^*$ for all $f \in Y^*$.*
3. *T is continuous $(X, \mathrm{wk}) \to (Y, \mathrm{wk})$.*

Proof. (1) $\implies$ (2): This is because T is continuous.

(2) $\implies$ (3): Consider

$$\begin{aligned} T^{-1}\left[\bigcap_{i=1}^m \{y : |\langle f_i, y\rangle| < \varepsilon_i\}\right] &= \bigcap_{i=1}^n \{x : |\langle f_i, Tx\rangle| < \varepsilon_i\} \\ &= \bigcap_{i=1}^n \{x : |\langle f_i \circ T, x\rangle| < \varepsilon_i\} \end{aligned}$$

(3) $\implies$ (1): We must show that $T[B_X] \subseteq MB_Y$ for some $M < \infty$. Given $f \in X^*$, consider

$$f[T[B_X]] = \{f(Tx) : \|x\|_X \leq 1\}$$

We know that there is a weak neighborhood $U \ni 0_X$ such that $T[U] \subseteq \{y : |f(y)| < 1\}$. The weak topology is weaker than the norm topology, so there exists some $\varepsilon > 0$ such that $T[B_X] \subseteq \{y : |f(y)| < 1/\varepsilon\}$. So $|f[T[B_X]]| \leq 1/\varepsilon$. □

Proposition 1.3. *Adjoints have the following properties:*

1. $(\alpha A + \beta B)^* = \alpha A^* + \beta B^*$ *for all* $A, B \in \mathcal{B}(X, Y)$.

2. *If* $T \in \mathcal{B}(X, Y)$, *then* T^* *is continuous from* (Y^*, wk^*) *to* (X^*, wk^*).

Remark 1.1. Riesz representation gives $H^* \cong H$ via $L_h := \langle \cdot, h\rangle \mapsto h$, which is conjugate-linear in h. So $L_{\alpha h} = \overline{\alpha} \cdot L_h$.

If $H = \mathbb{C}^n$, then A is represntaed by $[a_{i,j}] \in M_{n,n}(\mathbb{C})$. Then $H^* \cong \mathbb{C}^n$, so A^* is represented by $[a_{j,i}]$. But A^* on H itself is represented by $[\overline{a}_{j,i}]$.

Proposition 1.4. *Let* X, Y *be Banach, and let* $A \in \mathcal{B}(X, Y)$.

1. $A^{**}|_X = A$.

2. $\|A^*\| = \|A\|$.

3. *If* A *is invertible, so is* A^*, *and* $(A^*)^{-1} = (A^{-1})^*$.

4. *If* $B \in \mathcal{B}(Y, Z)$, *then* $(BA)^* = A^*B^*$.

Proof. For (2), we need to show that $\|A^*\| \geq \|A\|$. We know that $\|A^{**}\| \leq \|A^*\| \leq \|A\|$. Since A^{**} is an extension of A to a larger space, $\|A^{**}\| \geq \|A\|$. So these are all equal. □

Example 1.1. Let $1 < p, p' < \infty$. Consider an operator $L^p(\mu) \to \mu^{p'}(\nu)$ given by

$$Tf(y) = \int f(x)K(x, y)\, d\mu(x).$$

Then $T^* : L^{q'}(\nu) \to L^q(\mu)$. For $g \in L^{q'}(\nu)$ and $f \in L^p(\mu)$,

$$\langle T^*g, f\rangle = \langle g, Tf\rangle = \iint g(y)f(x)K(x, y)\, d\mu(x)\, d\nu(y).$$

So

$$T^*g(y) = \int (y)K(y, x)\, d\nu(y),$$

If we are in a Hilbert space, we may want to do $\langle f, g\rangle = \int f\overline{g}\, d\mu$ instead.

Proposition 1.5. *Let* $A \in \mathcal{B}(X, Y)$*. Then* $\ker A^* = (\operatorname{ran} A)^\perp$*, and* $\ker A = {}^\perp(\operatorname{ran} A^*)$*.*

Proof. We prove the second one; the first is similar. We have

$$\begin{aligned} x \in \ker A &\iff Ax = 0 \\ &\iff \langle Ax, y^* \rangle = 0 \qquad \forall y^* \in Y^* \\ &\iff \langle x, A^* y^* \rangle = 0 \qquad \forall y^* \in Y^* \\ &\iff x \in {}^\perp(\operatorname{ran} A^*). \end{aligned}$$

□

Proposition 1.6. *Let* $A \in \mathcal{B}(X, Y)$*. Then* A *is invertible if and only if* A^* *is invertible.*

Proof. ($\Longleftarrow$): If $\ker A^* = 0$, then $\operatorname{ran} A$ is dense. If $\operatorname{ran} A^* = X^*$, then $\ker A = \{0\}$. To finish, we need to show that $\operatorname{ran} A$ is closed. This follows because if $y = Ax$, then

$$\begin{aligned} \|Ax\| &= \sup\{|f(Ax)| : \|f\|_{Y^*} \leq 1\} \\ &= \sup\{|A^* f(x)| : \|f\|_{Y^*} \leq 1\} \\ &= \sup\{|g(x)| : f \in A^*[B_{Y^*}]\} \end{aligned}$$

For some $c > 0$,

$$\begin{aligned} &\geq \sup\{|g(x)| : g \in cB_{X^*}\} \\ &= c\|x\|_X. \end{aligned}$$

So $\operatorname{ran} A$ is closed. □

1.2 The Banach-Stone theorem

Example 1.2. Let X, Y be compact, Hausdorff spaces, let $\tau : Y \to X$ be a homeomorphism, and let $\alpha : Y \to S^1$ be continuous. Define $T : C(X) \to C(Y)$ by $Tf(y) = \alpha(y) \cdot f(\tau(y))$. Then T is an isometric isomorphism.

Theorem 1.1 (Banach-Stone)**.** *Any isometric isomorphism* $C(X) \to C(Y)$ *is of this form.*

The key is to tell you what Banach space structure of $C(X)$ to look at to recover what X is.

We know that T^* is an isometric isomorphism from $M(Y) \to M(X)$.

Proposition 1.7. *Let* X *be a compact, Hausdorff space.*

1. *Let* $X \times S^1 \to M(X)$ *send* $(x, \alpha) \mapsto x \cdot \delta_x$*. This is a homeomorphism from* $X \times S^1$ *to* $(\operatorname{ext}(B_{M(X)}), \text{wk*})$*.*
2. *Let* $X \times \{1\} \to M(X)$ *send* $x \mapsto \delta_x$*. This is a homeomorphism* $X \to \operatorname{ext} P(X)$*.*

Proof. We prove (1). We must show that $\mu \in B_{M(X)}$ is extreme if and only if $\mu = \alpha\delta_x$ for some α, x. □