# Math 255B Lecture Notes
# Functional Analysis

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# 1 Fredholm Theory

## 1.1 Fredholm operators

**Definition 1.1.** Let $B_1, B_2$ be Banach spaces. An operator $T \in \mathcal{L}(B_1, B_2)$ is called **Fredholm** if the kernel $\ker T = \{x \in B_1 : Tx = 0\}$ and the cokernel $\operatorname{coker} T = B_2 / \operatorname{im} T$ are finite-dimensional. We define the **index** if $T$ to be $\operatorname{ind} T = \dim \ker T - \dim \operatorname{coker} T \in \mathbb{Z}$.

**Remark 1.1.** If $T \in \mathcal{L}(B_1, B_2)$, then $\ker T$ is a closed subspace of $B_1$. However, $\operatorname{im} T$ need not necessarily be closed: take $B_1 = B_2 = C([0,1])$ and $(Tf)(x) = \int_0^x f(y)\, dy$.

So this is an algebraic condition. However, this implies an analytic condition on $T$:

**Proposition 1.1.** *If $T \in \mathcal{L}(B_1, B_2)$ and $\dim \operatorname{coker} T < \infty$, then $\operatorname{im} T$ is closed.*

*Proof.* We may assume $T$ is injective, for otherwise, we can consider $\widetilde{T} : B_1 / \ker T \to B_2$ sending $x + \ker T \mapsto Tx$; then $\operatorname{im} \widetilde{T} = \operatorname{im} T$, and $\widetilde{T}$ is injective. Let $\dim \operatorname{coker} T = n < \infty$, and let $x_1, \dots, x_n \in B_2$ be such that $x_1 + \operatorname{im} T, \dots, x_n + \operatorname{im} T$ form a basis for $\operatorname{coker} T$. Let $S : \mathbb{C}^n \to B_2$ send $(a_1, \dots, a_n) \mapsto \sum_{j=1}^n a_j x_j$. Then $S$ is injective, and $B_2 = \operatorname{im} T \oplus \operatorname{im} S$. It follows that $T_1 : B_1 \oplus \mathbb{C}^n \to B_2$ sending $(x, a) \mapsto Tx + Sa$ is a bijection. By the open mapping theorem, $T_1$ is a linear homeomorphism. Then $\operatorname{im} T = T_1(B_1 \oplus \{0\}) \subseteq B_2$ is closed. $\square$

## 1.2 Behavior of the index under perturbation

If $\dim B_j < \infty$ for $j = 1, 2$, then

$$\operatorname{ind} T = \dim \ker T - (\dim B_2 - \dim \operatorname{im} T) = \dim B_1 - \dim B_2.$$

Remarkably, for Fredholm operators, this property also extends to a similar property in the infinite dimensional case.

**Theorem 1.1.** *Let $T \in \mathcal{L}(B_1, B_2)$ be a Fredholm operator. If $S \in \mathcal{L}(B_1, B_2)$ is such that $\|S\|_{\mathcal{L}(B_1, B_2)}$ is sufficiently small, then $T + S$ is Fredholm, and $\operatorname{ind}(T + S) = \operatorname{ind} T$.*

To prove this, we have a lemma.

**Lemma 1.1.** *Let $B$ be a Banach space, and let $S \in \mathcal{L}(B, B)$ be such that $\|S\| < 1$. Then $1 - S$ has an inverse (so $\operatorname{ind}(1 - S) = 0$).*

*Proof.* The Neumann series $R = \sum_{k=0}^\infty S^k$ converges in $\mathcal{L}(B, B)$, and $R(1 - S) = (1 - S)R = 1$. $\square$

**Remark 1.2.** If $T \in \mathcal{L}(B_1, B_2)$ is invertible and $\|S\|$ is small, then $T + S$ is invertible: $T + S = T(1 + T^{-1}S)$ is invertible if $\|S\| < 1/\|T^{-1}\|$.

To prove the theorem, we will reduce to this case.

*Proof.* Write $n_+ = \dim\ker T$ and $n_- = \dim\operatorname{coker} T$. Let $R_- : \mathbb{C}^{n_-} \to B_2$ be injective and such that $B_2 = \operatorname{im} T \oplus R_-(\mathbb{C}^{n_-})$ (as we have constructed before). Let $e_1, \ldots, e_{n_+}$ be a basis for $\ker T$, and let $\varphi_1, \ldots, \varphi_{n_+} \in B_1^*$ be such that

$$\varphi_j(e_k) = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$$

for all $j, k$; such continuous, linear forms exist by Hahn-Banach. Let $R_+ : B_1 \to \mathbb{C}^{n_+}$ send $x \mapsto (\varphi_1(x), \ldots, \varphi_{n_+}(x))$. Then $R_+$ is surjective, and $R_+|_{\ker T}$ is bijective.

Let us introduce the **Grushin operator**[1]

$$\mathcal{P} = \begin{bmatrix} T & R_- \\ R_+ & 0 \end{bmatrix} : B_1 \oplus \mathbb{C}^{n_-} \to B_2 \oplus \mathbb{C}^{n_+}.$$

We claim that $\mathcal{P}$ is invertible: If $\mathcal{P}\begin{bmatrix} x \\ a_- \end{bmatrix} = 0$, then $Tx + R_- a_- = 0$ and $R_+ x = 0$. Then $a_- = 0$, so $x \in \ker T$. Since $R_+$ is bijective on $\ker T$, we get $x = 0$. For surjectivity, we want to solve $Tx + R_- a_- = y$ and $R_+ x = b$. Write $y = Tz + R_- c_-$. Then $a_- = c_-$ and $x - z \in \ker T$, so $x = z + \sum \alpha_j e_j$. We can take $\alpha_j = b_j - \varphi_j(z)$ for $1 \leq j \leq n_+$.

If $\|S\|$ is small enough, then

$$\widetilde{\mathcal{P}} = \begin{bmatrix} T + S & R_- \\ R_+ & 0 \end{bmatrix}$$

is invertible, and we introduce the inverse

$$\mathcal{E} = \begin{bmatrix} E & E_+ \\ E_- & E_{-+} \end{bmatrix} : B_2 \oplus \mathbb{C}^{n_+} \to B_1 \oplus \mathbb{C}^{n_-}.$$

We will finish the proof next time. □

[1] This terminology is not necessarily standard.

# 2 Perturbation of Fredholm Operators and The Logarithmic Law

## 2.1 Perturbation of Fredholm operators

Last time, said that $T \in \mathcal{L}(B_1, B_2)$ is **Fredholm** if $\dim \ker T < \infty$ and $\dim B_2 / \operatorname{im} T < \infty$. We were proving that the Fredholm property is preserved under small perturbations.

**Theorem 2.1.** *Let $T \in \mathcal{L}(B_1, B_2)$ be a Fredholm operator. If $S \in \mathcal{L}(B_1, B_2)$ is such that $\|S\|_{\mathcal{L}(B_1,B_2)}$ is sufficiently small, then $T + S$ is Fredholm, and* $\operatorname{ind}(T+S) = \operatorname{ind} T$.

*Proof.* Produce

$$\mathcal{P} = \begin{bmatrix} T & R_- \\ R_+ & 0 \end{bmatrix} : B_1 \oplus \mathbb{C}^{n_-} \to B_2 \oplus \mathbb{C}^{n_+},$$

where $n_+ = \dim \ker T$ and $n_- = \dim \operatorname{coker} T$. We get that

$$\widetilde{\mathcal{P}} = \begin{bmatrix} T+S & R_- \\ R_+ & 0 \end{bmatrix}$$

is invertible as well, with inverse

$$\mathcal{E} = \begin{bmatrix} E & E_+ \\ E_- & E_{-+} \end{bmatrix} : B_2 \oplus \mathbb{C}^{n_+} \to B_1 \oplus \mathbb{C}^{n_-},$$

where

$$E : B_2 \to B_1, \qquad E_+ : \mathbb{C}^{n_+} \to B_1, \qquad E_- : B_2 \to \mathbb{C}^{n_-}, \qquad E_{-+} : \mathbb{C}^{n_+} \to \mathbb{C}^{n_-}.$$

Since $\mathcal{E}$ is a right inverse for $\widetilde{\mathcal{P}}$,

$$\begin{bmatrix} T+S & R_- \\ R_+ & 0 \end{bmatrix} \begin{bmatrix} E & E_+ \\ E_- & E_{-+} \end{bmatrix} = \begin{bmatrix} * & * \\ * & R_+E_+ \end{bmatrix}.$$

This is the identity map, so $R_+E_+ = 1$ on $\mathbb{C}^{n_+}$. So $E_+$ is injective. Similarly, we get $E_-R_- = 1$ on $\mathbb{C}^{n_-}$, so $E_-$ is surjective.

We claim that $T + S$ is Fredholm. For the kernel, we have $x \in \ker(T+S) \iff (T+S)x = 0$. We can write this as

$$\widetilde{\mathcal{P}} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ a_+ \end{bmatrix},$$

where $a_= = R_+x \in \mathbb{C}^{n_+}$. Using the inverse $\mathcal{E}$, this is

$$\begin{bmatrix} x \\ 0 \end{bmatrix} = \mathcal{E} \begin{bmatrix} 0 \\ a_+ \end{bmatrix} = \begin{bmatrix} E_+a_+ \\ E_{-+}a_+ \end{bmatrix}.$$

So we get that $x \in \ker(T+S) \iff x = E_+ a_+$ for some $a_+ \in \ker E_+$. So $E_+ : \ker E_{-+} \to \ker(T+S)$ is surjective. Since we already know $E_+$ is injective, we get that $\ker(T+S)$ is finite dimensional with $\dim\ker(T+S) = \dim\ker(E_{-+}) \le n_+$.

Next consider $B_2/\operatorname{im}(T+S)$: Given $y$,

$$\begin{aligned}(T+S)x = y &\iff \widetilde{\mathcal{P}}\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} y \\ a_+ \end{bmatrix} \\ &\iff \begin{bmatrix} x \\ 0 \end{bmatrix} = \mathcal{E}\begin{bmatrix} y \\ a_+ \end{bmatrix}.\end{aligned}$$

So we get that $x = Ey + E_+a_+$ and $0 = E_-y + E_{-+}a_+$. We get that $y \in \operatorname{im}(T+S) \iff E_-y \in \operatorname{im} E_{-+}$. Now consider $B_2/\operatorname{im}(T+S) \to \mathbb{C}^{n_-}/E_{-+}$ sending $y + \operatorname{im}(T+S) \mapsto E_-y + \operatorname{im} E_{-+}$. This map is surjective, as $E_-$ is surjective, and it is also injective. So $\dim\operatorname{coker}(T+S) = \dim\operatorname{coker} E_{-+} < \infty$.

So $T+S$ is Fredholm, and

$$\operatorname{ind}(T+S) = \dim\ker E_{-+} - \dim\operatorname{coker} E_{-+} = \operatorname{ind} E_{-+} = n_+ - n_- = \operatorname{ind}(T). \qquad \square$$

**Corollary 2.1.** *The set of Fredholm operators is open in $\mathcal{L}(B_1, B_2)$, and $T \mapsto \operatorname{ind}(T)$ is locally constant.*

The proof also gives the following:

**Corollary 2.2.** *$T \mapsto \dim\ker T$ is upper-semicontinuous on the set of Fredholm operators.*

## 2.2 The logarithmic law

**Proposition 2.1.** *Let $T_1 \in \mathcal{L}(B_1, B_2)$ and $T_2 \in \mathcal{L}(B_2, B_3)$ be Fredholm. Then $T_2T_1$ is Fredholm, and we have the **logarithmic law**:*

$$\operatorname{ind} T_2T_1 = \operatorname{ind} T_2 + \operatorname{ind} T_1.$$

*Proof.* Consider $T_1' : \ker T_2T_1 \to \ker T_2$ sending $x \mapsto T_1x$. Then $\ker T_1' = \ker T_1$, so $\dim(\ker(T_2T_1)/\ker T_1) \le \dim\ker T_2$. So $\dim\ker T_2T_1 < \infty$.

Now consider

$$0 \longrightarrow B_2/\operatorname{im} T_1 \xrightarrow{T_2'} B_3/\operatorname{im} T_2T_1 \xrightarrow{q} B_3/\operatorname{im} T_2 \longrightarrow 0,$$

where

$$T_2'(x + \operatorname{im} T_1) = T_2x + \operatorname{im} T_2T_1, \qquad q(x + \operatorname{im} T_2T_1) = x + \operatorname{im} T_2.$$

The sequence is exact at $B_3/\operatorname{im} T_2T_1$: $\operatorname{im} T_2' \subseteq \ker q$ by definition, and if $x + \operatorname{im} T_2T_1 \in \ker q$, then $x \in \operatorname{im} T_2$, so $x + \operatorname{im} T_2T_1 \in \operatorname{im} T_2'$. We have

$$\dim(\operatorname{coker}(T_2T_1)/\ker q) \le \dim\operatorname{coker} T_2, \qquad \dim\ker q = \dim\operatorname{im} T_2' \le \dim\operatorname{coker} T_1,$$

so

$$\dim \operatorname{coker}(T_2T_1) \leq \dim \operatorname{coker} T_1 + \dim \operatorname{coker} T_2.$$

To compute the index, consider

$$L(t) = \begin{bmatrix} I_2 & 0 \\ 0 & T_2 \end{bmatrix} \begin{bmatrix} I_2 \cos t & I_2 \sin t \\ -I_2 \sin t & I_2 \cos t \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ 0 & I_2 \end{bmatrix} : B_1 \oplus B_2 \to B_2 \oplus B_3, \qquad t \in \mathbb{R}, I_2 = \mathrm{id}_{B_2}.$$

This is a product of three Fredholm operators, so $L(t)$ is Fredholm for all $t$ and $t \mapsto \mathcal{L}(t)$ is continuous. So $\operatorname{ind} L(t)$ is independent of $t$! When $t = 0$,

$$L(0) = \begin{bmatrix} I_2 & 0 \\ 0 & T_2 \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ 0 & I_2 \end{bmatrix} = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix},$$

so $\operatorname{ind} L(0) = \operatorname{ind} T_1 + \operatorname{ind} T_2$. When $t = -\pi/2$, we get

$$L(-\pi/2) = \begin{bmatrix} 0 & -I_2 \\ T_2T_1 & 0 \end{bmatrix}.$$

So

$$L(-\pi/2) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ T_2T_1x \end{bmatrix},$$

which gives $\ker L(-\pi/2) = \ker T_2T_1 \oplus \{0\}$. We get $\operatorname{ind} L(-\pi/2) = \operatorname{ind}(T_2T_1)$. Since the index is locally constant, we get the logarithmic law. $\square$

# 3 The Fredholm-Riesz Theorem

## 3.1 The Fredholm-Riesz theorem

**Theorem 3.1** (Fredholm-Riesz)**.** *Let $B$ be a Banach space, and let $T \in \mathcal{L}(B, B)$ be compact. Then $1 - T$ is Fredholm, and* $\operatorname{ind}(1 - T) = 0$.

**Remark 3.1.** If $B$ is a Hilbert space, we can prove this more easily by using the fact that compact operators can be approximated by finite rank operators.

**Proposition 3.1.** *Let $T \in \mathcal{L}(B, B)$ be compact. Then*

1. $\ker(1 - T)$ *is finite dimensional.*

2. $\operatorname{im}(1 - T)$ *is closed.*

*Proof.* 1. Let $x_n \in \ker(1 - T)$ with $\|x_n\| \leq 1$. Then $x_n = Tx_n$ has a convergent subsequence. Then the identity map on $\ker(1-T)$ is compact, so $\dim \ker(1-T) < \infty$ (by Riesz's theorem).

2. Let $y \in \overline{\operatorname{im}(1 - T)}$, and let $x_n \in B$ be such that $y_n = (1 - T)x_n \to y$. Consider $\operatorname{dist}(x_n, \ker(1 - T)) = \inf_{z \in \ker(1-T)} \|x_n - z\|$. There exists some $z_n \in \ker(1 - T)$ realizing this infimum: $\|x_n - z_n\| = \operatorname{dist}(x_n, \ker(1 - T))$.

   We claim that the sequence $(x_n - z_n)$ is bounded: otherwise, $\|x_n - z_n\| \to \infty$ along a subsequence. Let $w_n = \frac{x_n - z_n}{\|x_n - z_n\|}$, so

   $$(1 - T)w_n = \frac{(1 - T)(x_n - z_n)}{\|x_n - z_n\|} = \frac{\eta_n}{\|x_n - z_n\|} \to 0.$$

   Passing to a subsequence, we may assume that $Tw_n \to v \in B$ and then $w_n \to v$, where $v \in \ker(1 - T)$. Now

   $$\operatorname{dist}(w_n, \ker(1 - T)) = \inf_{z \in \ker(1-T)} \frac{\|x_n - z_n - z\|}{\|x_n - z_n\|} = \frac{\operatorname{dist}(x_n, \ker(1 - T))}{\|x_n - z_n\|} = 1$$

   for all $n$. This proves the claim.

   Passing to a subsequence, we may assume that $T(x_n - z_n) \to \ell \in B$. Also, $y_n = (1 - T)(x_n - z_n) \to y$, so $x_n - z_n \to y + \ell = g$. Since $T$ is continuous, $(1 - T)g = \lim_{n\to\infty}(1 - T)(x_n - z_n) = y$. So $y \in \operatorname{im}(1 - T)$. □

## 3.2 Adjoints of inclusions and quotients

To show that $\dim \operatorname{coker} < \infty$, we will use duality arguments:

**Definition 3.1.** If $B_1, B_2$ are Banach spaces with duals $B_1^*, B_2^*$ and bilinear maps $\langle x, \xi \rangle_j : B_j \times B_j^* \to \mathbb{C}$ and if $T \in \mathcal{L}(B_1, B_2)$, then the **adjoint** $T^* \mathcal{L}(B_2^*, B_1^*)$ is defined by

$$\langle Tx, \eta \rangle_2 = \langle x, T^* \eta \rangle_1 \qquad \forall x \in B_1, \eta \in B_2^*.$$

**Definition 3.2.** If $B$ is a Banach space and $W \subseteq B$ is a closed subspace, the **annihilator** $W^o \subseteq B^*$ is given by

$$W^o = \{\xi \in B^* : \langle x, \xi \rangle = 0 \, \forall x \in W\}.$$

**Proposition 3.2.** *Let $B$ be a Banach space, and let $W \subseteq B$ be a closed subspace.*

1. *Let $i : W \to B$ be the inclusion map. Then $i^* : B^* \to W^*$ vanishes on $W^o$ and induces an isometric bijection $B^*/W^o \to W^*$.*

2. *Let $q : B \to B/W$ be the quotient map. Then the adjoint $q^* : (B/W)^* \to B^*$ is an isometry with the range $W^o$.*

*Proof.* 1. We have $\langle ix, \xi \rangle = \langle x, i^* \xi \rangle$, so $i^* \xi$ is the restriction of $\xi$ to $W$. So $\ker i^* = W^o$. $i^* : B^* \to W^*$ is surjective by Hahn-Banach.

2. We have $\langle qx, \eta \rangle = \langle x, q^* \eta \rangle$, so $q^* : (B/W)^* \to B^*$ sends $q^* \eta$ to $x \mapsto \langle qx, \eta \rangle$. So if $q^* \eta = 0$, then $\eta = 0$; i.e. $q^*$ is injective. Also, $\operatorname{im} q^* \subseteq W^o$, and in fact, $\operatorname{im} q^* = W^o$: If $\xi \in W^o$, define $\eta$ by $\langle qx, \eta \rangle = \langle x, \xi \rangle$ and $\xi = q^* \eta$. Check that the norms are equal. $\square$

### 3.3 Proof of the Fredholm-Riesz theorem

Recall that $T \in \mathcal{L}(B, B)$ is compact. We want to show that $\operatorname{coker}(1 - T)$ is finite dimensional, and we know that it is closed.

*Proof.* Apply $(B/W)^* \cong W^o$ with $W = \operatorname{im}(1 - T)$.

$$(\operatorname{im}(1 - T))^o = \{\xi \in B^* : \langle (1 - T)x, \xi \rangle = 0 \, \forall x \in B\} = \ker(1 - T^*).$$

$T^*$ is compact, so $\dim(\operatorname{im}(1 - T))^o < \infty$. This shows that $(\operatorname{coker}(1 - T))^* \cong \ker(1 - T^*)$, so $\dim \operatorname{coker}(1 - T) = \dim \ker(1 - T^*) < \infty$. So $1 - T$ is Fredholm.

Finally, for $0 \leq t \leq 1$,

$$\operatorname{ind}(1 - T) = \operatorname{ind}(1 - tT) = \operatorname{ind} 1 = 0. \qquad \square$$

# 4 Sums of Fredholm and Compact Operators and The Toeplitz Index Theorem

## 4.1 Fredholm plus compact is Fredholm

Last time, we proved the Riesz-Fredholm theorem, which says that if $T \in \mathcal{L}(B, B)$ is compact, then $1 + T$ is a Fredholm operator with $\operatorname{ind}(1 + T) = 0$.

**Proposition 4.1.** *An operator $T \in \mathcal{L}(B_1, B_2)$ is Fredholm if and only if there exists a map $S \in \mathcal{L}(B_2, B_1)$ such that $TS - 1$ and $ST - 1$ are compact on $B_2$ and $B_1$, respectively.*

*Proof.* ( $\Longleftarrow$ ): Let $S \in \mathcal{L}(B_2, B_1)$ be such that $ST = 1 + K_1$ and $TS = 1 + K_2$, where $K_j$ is compact on $B_j$ for $j = 1, 2$. Then $\ker T \subseteq \ker(1 + K_1)$, so $\ker T$ is finite-dimensional. On the other hand, $\operatorname{im} T \supseteq \operatorname{im}(1 + K_2)$: Let $Y \subseteq B_2$ be such that $\dim Y = \dim \operatorname{coker}(1 + K_2)$, so $B_2 = \operatorname{im}(1 + K_2) \oplus Y$. If $Y = \operatorname{im} T \cap Y$, so $Y = Y_1 \oplus Y_2$, then $B_2 = \operatorname{im} T \oplus Y_2$. So we get $\dim \operatorname{coker} T = \dim Y_2 \leq \dim Y = \dim \operatorname{coker}(1 + K_2) < \infty$.

( $\Longrightarrow$ ): We follow the Grushin approach: If $n_+ = \dim \ker T$ and $n_- = \dim \operatorname{coker} T$, then there exist an injective $R_- : \mathbb{C}^{n_-} \to B_2$ and a surjective $R_+ : B_1 \to \mathbb{C}^{n_+}$ such that the Grushin operator

$$\mathcal{P} = \begin{bmatrix} T & R_- \\ R_+ & 0 \end{bmatrix} : B_1 \oplus \mathbb{C}^{n_-} \to B_2 \oplus \mathbb{C}^{n_+}$$

is invertible with inverse

$$\mathcal{E} = \begin{bmatrix} E & E_+ \\ E_- & E_{-+} \end{bmatrix}.$$

Moreover,

$$1 = \mathcal{P}\mathcal{E} = \begin{bmatrix} R & R_- \\ R_+ & 0 \end{bmatrix} \begin{bmatrix} E & E_+ \\ E_- & E_{-+} \end{bmatrix} = \begin{bmatrix} TE + R_-E_- & * \\ * & * \end{bmatrix},$$

so $TE + R_-E_- = 1$ on $B_2$, where $R_-E_-$ has finite rank. Similarly, using $1 = \mathcal{E}\mathcal{P}$, we get $ET - 1 = -E_+R_+$, where $E_+R_+$ has finite rank on $B_1$. □

**Remark 4.1.** If $S \in \mathcal{L}(B_2, B_1)$ is such that $ST - 1$ and $TS - 1$ are compact, then $S$ is Fredholm, and $\operatorname{ind}(ST) = 0$. The logarithmic law gives $\operatorname{ind}(ST) = \operatorname{ind} S + \operatorname{ind} T$, so we get $\operatorname{ind} S = -\operatorname{ind} T$.

**Theorem 4.1** (Fredholm theory)**.** *Let $T \in \mathcal{L}(B_1, B_2)$ be Fredholm, and let $S \in \mathcal{L}(B_1, B_2)$ be compact. Then $T + S$ is Fredholm, and* $\operatorname{ind}(T + S) = \operatorname{ind} T$.

*Proof.* Let $E \in \mathcal{L}(B_2, B_1)$ be such that $TE - 1, ET - 1$ are compact. Then $(T + S)E - 1$ and $S(T + S) - 1$ are compact, so $T + S$ is Fredholm. Moreover, $\operatorname{ind}(T + S) = \operatorname{ind}(T + tS) = \operatorname{ind} T$ for all $t \in [0, 1]$. □

## 4.2 The Toeplitz index theorem

Here is a nice example of a Fredholm operator.

**Example 4.1.** Consider $L^2((0,2\pi)) \cong L^2(\mathbb{R}/2\pi\mathbb{Z})$. If $u \in L^2((0,2\pi))$ and the Fourier coefficients are $\widehat{u}(n) = \frac{1}{2\pi}\int_0^{2\pi} u(\theta)e^{-in\theta}\,d\theta$, then $u(\theta) \sim \sum_{n\in\mathbb{Z}} \widehat{u}(n)e^{in\theta}$. Consider the **Hardy space** $H = \{u \in L^2 : \widehat{u}(n) = 0 \text{ for } n < 0\}$, which is a closed subspace of $L^2((0,2\pi))$. The associated orthogonal projection $\pi : L^2 \to H$ sends $\sum_{n\in\mathbb{Z}} \widehat{u}(n)e^{in\theta} \mapsto \sum_{n\geq 0} \widehat{u}(n)e^{in\theta}$. Let $f \in L^\infty((0,2\pi))$. Associated to $f$ is the **Toeplitz operator** $\operatorname{Top}(f) : H \to H$ given by $\operatorname{Top}(f)u = \pi(fu)$. Then $\operatorname{Top}(f) \in \mathcal{L}(H,H)$, and $\|\operatorname{Top}(f)\|_{\mathcal{L}(H,H)} \leq \|f\|_\infty$.

**Theorem 4.2** (Toeplitz index theorem)**.** *If $f \in C(\mathbb{R}/2\pi\mathbb{Z})$ is nonvanishing, then* $\operatorname{Top}(f)$ *is Fredholm on $H$, and* $\operatorname{ind}\operatorname{Top}(f) = -$*winding number*$(f)$.

To define the winding number, write $f(\theta) = r(\theta)e^{i\varphi(\theta)}$, where $r > 0$ and $r, \varphi$ are continuous on $[0,2\pi]$. Then the winding number of $f$ is $\frac{\varphi(2\pi)-\varphi(0)}{2\pi}$.

*Proof.* To prove the Fredholm property of $\operatorname{Top}(f)$, we will try to invert $\operatorname{Top}(f)$ with a compact error. We claim that if $f, g \in C(\mathbb{R}/2\pi\mathbb{Z})$, then $\operatorname{Top}(f)\operatorname{Top}(f) = \operatorname{Top}(fg) + K$, where $K$ is compact. Write $\operatorname{Top}(f) = \pi M_f$ and $\operatorname{Top}(f) = \pi M_g$, where $M$ means a multiplication operator. Then

$$\pi M_f \pi M_g = \pi(\pi M_f + [M_f,\pi])M_g = \pi^2 M_{fg} + \pi[M_f,\pi]M_g = \operatorname{Top}(fg) + K,$$

where $[M_f,\pi] = M_f\pi - \pi M_f$ is the commutator $L^2 \to L^2$ and $K = \pi[M_f,\pi]M_g$. To show that $K$ is compact, it suffices to show that $[M_f,\pi]$ is compact on $L^2$.

Case 1: If $f(\theta) = e^{in\theta}$, with $n \in \mathbb{Z}$, then

$$[M_{e^{in\theta}}\pi]e^{ik\theta} = (e^{in\theta} \circ \pi - \pi \circ e^{in\theta})e^{ik\theta}$$

If $n > 0$,

$$= \begin{cases} 0 & k \geq 0 \\ -\pi(e^{i(n+k)\theta}) & k < 0, \end{cases}$$

where the latter expression $= 0$ if $k < -n$. So $[M_{e^{in\theta}},\pi]$ is of finite rank on $L^2$.

We will finish the proof next time. $\square$

# 5 The Toeplitz Index Theorem and Analytic Fredholm Theory

## 5.1 The Toeplitz index theorem

Last time, we had the Hardy space $H \subseteq L^2(\mathbb{R}/2\pi\mathbb{Z})$ of functions $u$ with $\widehat{u}(n) = 0$ for $n < 0$. Given $f \in C(\mathbb{R}/2\pi\mathbb{Z})$, we defined $\operatorname{Top}(f) = \pi M_f$.

**Theorem 5.1** (Toeplitz index theorem)**.** *If $f \in C(\mathbb{R}/2\pi\mathbb{Z})$ is nonvanishing, then* $\operatorname{Top}(f)$ *is Fredholm on $H$, and* $\operatorname{ind}\operatorname{Top}(f) = -$*winding number*$(f)$.

*Proof.* We had the claim that for all $f, g \in C(\mathbb{R}/2\pi\mathbb{Z})$, then $\operatorname{Top}(f)\operatorname{Top}(g) - \operatorname{Top}(fg)$ is compact. We saw that this is $\pi[M_f, \pi]M_g$, so we only need to show that $[M_f, \pi]$ is compact from $L^2 \to L^2$. If $f(\theta) = e^{in\theta}$ (or more generally, a trigonometric polynomial), then $[M_f, \pi]$ is of finite rank; we showed this last time.

In general, given $f \in C(\mathbb{R}/2\pi\mathbb{Z})$, let $f_n$ be trigonometric polynomials such that $f_n \to f$ uniformly on $\mathbb{R}/2\pi\mathbb{Z}$. Then

$$\|[M_f, \pi] - [M_{f_n}, \pi]\| = \|[M_{f-f_n}, \pi]\| \leq 2\|f - f_n\|_u \to 0.$$

So $[M_f, \pi]$ is compact, and we get the claim.

If $f \neq 0$, we take $g = 1/f$, so $\operatorname{Top}(f)\operatorname{Top}(g) - I$ is compact. So $\operatorname{Top}(f)$ is Fredholm. To compute $\operatorname{ind}\operatorname{Top}(f)$, observe that if $g, h$ are continuous (and nonvanishing), then

$$\operatorname{ind}\operatorname{Top}(gh) = \operatorname{ind}(\operatorname{Top}(g)\operatorname{Top}(h)) = \operatorname{ind}\operatorname{Top}(g) + \operatorname{ind}\operatorname{Top}(f).$$

Write $f(\theta) = r(\theta)e^{-\varphi(\theta)}$ with $r, \varphi$ continuous on $[0, 2\pi]$ and $r > 0$. Then

$$\operatorname{ind}\operatorname{Top}(f) = \operatorname{ind}\operatorname{Top}(r) + \operatorname{ind}\operatorname{Top}(e^{i\varphi})$$

We have $\operatorname{ind}\operatorname{Top}(r) = \operatorname{ind}\operatorname{Top}(r_t)$ for $0 \leq t \leq 1$, where $r_t(\theta) = (1-t)r(\theta) + t1 > 0$. So $\operatorname{ind}\operatorname{Top}(r) = 0$.

$$= \operatorname{ind}\operatorname{Top}(e^{i\varphi}).$$

To compute $\operatorname{ind}\operatorname{Top}(e^{i\varphi})$, consider $f_t(\theta) = e^{(1-t)i\varphi(\theta) + iNt\theta}$ for $0 \leq t \leq 1$, where $N = \frac{\varphi(2\pi) - \varphi(0)}{2\pi}$ is the winding number. Then $f_t$ is $2\pi$-periodic and continuous in $t$. We get

$$\begin{aligned} \operatorname{ind}\operatorname{Top}(e^{i\varphi}) &= \operatorname{ind}\operatorname{Top}(f_t) \\ &= \operatorname{ind}\operatorname{Top}(e^{iN\theta}) \end{aligned}$$

In general, if $T$ is Fredholm, $\operatorname{ind} T = \dim\ker T - \dim\ker T^*$.

$$= \dim\ker\operatorname{Top}(e^{iN\theta}) - \dim\ker\operatorname{Top}(e^{iN\theta})^*$$

To find the adjoint, we have $\langle \mathrm{Top}(f)u, v\rangle_{L^2} = \langle \pi(fu), v\rangle_{L^2} = \langle fu, v\rangle_{L^2} = \left\langle u, \overline{f}v\right\rangle_{L^2} = \left\langle \pi u, \overline{f}v\right\rangle_{L^2} = \left\langle u, \mathrm{Top}(\overline{f})v\right\rangle$. So $\mathrm{Top}(f)^* = \mathrm{Top}(\overline{f})$.

$$= \dim\ker \mathrm{Top}(e^{iN\theta}) - \dim\ker \mathrm{Top}(e^{-iN\theta}).$$

Here, we have

$$\dim\ker \mathrm{Top}(e^{iN\theta}) = \begin{cases} 0 & N \geq 0 \\ -N & N < 0 \end{cases}.$$

Altogether, we get

$$\operatorname{ind} \mathrm{Top}(f) = -N. \qquad \square$$

## 5.2 Analytic Fredholm Theory

**Definition 5.1.** Let $\Omega \subseteq \mathbb{C}$. A **holomorphic family** $T(z) \in \mathcal{L}(B_1, B_2)$ for $z \in \Omega$ is a family such that $\Omega \to \mathcal{L}(B_1, B_2)$ sending $z \mapsto T(z)$ is holomorphic (as an operator-valued function).

**Remark 5.1.** We can define holomorphic operator-valued functions in two ways: $z \mapsto T(z)$ is **holomorphic** if

1. For all $z \in \Omega$, $\|\frac{T(z+h)-T(z)}{h} - T'(z)\| \to 0$ as $h \to 0$ for some $T'(z) \in \mathcal{L}(B_1, B_2)$.
2. For every $x \in B_1$ and $\xi \in B_2^*$, $z \mapsto \langle T(z)x, \xi\rangle$ is holomorphic.

**Theorem 5.2** (analytic Fredholm theory)**.** *Let $\Omega \subseteq \mathbb{C}$ be open and connected, and let $T(z) \in \mathcal{L}(B_1, B_2)$ for $z \in \Omega$ be a holomorphic family of Fredholm operators. Assume that there exists a $z_0 \in \Omega$ such that $T(z_0) : B_1 \to B_2$ is bijective. Then the set*

$$\Sigma = \{z \in \Omega : T(z) \text{ is not bijective}\}$$

*is discrete.*

*Proof.* Notice first that $\operatorname{ind} T(z) = \operatorname{ind} T(z_0) = 0$ for all $z$. Let $z_1 \in \Omega$, and write $n_0(z_1) = \dim\ker T(z_1) = \dim\operatorname{coker} T(z_1)$. Introduce the Grushin operator

$$\mathcal{P}_{z_1}(z) = \begin{bmatrix} T(z) & R_-(z_1) \\ R_+(z_1) & 0 \end{bmatrix} : B_1 \oplus \mathbb{C}^{n_0(z_1)} \to B_2 \oplus \mathbb{C}^{n_0(z_1)}.$$

We know that $\mathcal{P}_{z_1}(z_1)$ is invertible. So there is a connected neighborhood $N(z_1) \subseteq \Omega$ of $z_1$ such that $\mathcal{P}_{z_1}(z)$ is bijective for $z \in N(z_1)$, depending holomorphically on $z$. Let

$$\mathcal{E}_{z_1}(z) = \mathcal{P}_{z_1}(z)^{-1} : B_2 \oplus \mathbb{C}^{n_0(z_1)} \to B_1 \oplus \mathbb{C}^{n_0(z_1)}$$

$$\mathcal{E}_{z_1}(z) = \begin{bmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{bmatrix},$$

depending holomorphically on $z$.

We claim that for $z \in N(z_1)$, $T(z) : B_1 \to B_2$ is bijective $\iff$ $E_{-+}(z) : \mathbb{C}^{n_0} \to \mathbb{C}^{n_0}$ is bijective. This will allow us to analyze invertibility of $T(z)$ via a holomorphic function, $\det E_{-+}(z)$. $\square$

# 6 Consequences of Analytic Fredholm Theory

## 6.1 Analytic Fredholm theory

Last time, we were proving the analytic Fredholm theory.

**Theorem 6.1** (analytic Fredholm theory). *Let $\Omega \subseteq \mathbb{C}$ be open and connected, and let $T(z) \in \mathcal{L}(B_1, B_2)$ for $z \in \Omega$ be a holomorphic family of Fredholm operators. Assume that there exists a $z_0 \in \Omega$ such that $T(z_0) : B_1 \to B_2$ is bijective. Then the set*

$$\Sigma = \{z \in \Omega : T(z) \text{ is not bijective}\}$$

*is discrete.*

*Proof.* Let $z_1 \in \Omega$. Then there is a neighborhood $N(z_1)$ of $z_1$ such that for every $z \in N(z_1)$, the Grushin operator

$$\mathcal{P}_{z_1}(z) = \begin{bmatrix} T(z) & R_-(z) \\ R_+(z) & 0 \end{bmatrix}$$

is bijective with the inverse

$$\mathcal{E}_{z_1}(z) = \begin{bmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{bmatrix} : B_2 \oplus \mathbb{C}^{n_0} \to B_1 \oplus \mathbb{C}^{n_0}.$$

We claim that for $z \in N(z_1)$, $T(z) : B_1 \to B_2$ is bijective $\iff$ $E_{-+}(z) : \mathbb{C}^{n_0} \to \mathbb{C}^{n_0}$ is bijective.[2] Check:

$$\begin{bmatrix} T & R_- \\ R_+ & 0 \end{bmatrix} \begin{bmatrix} E & E_+ \\ E_- & E_{-+} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \implies TE + R_-E_- = 1, TE_+ + R_-E_{-+} = 0.$$

If $E_{-+}^{-1}$ exists, then $R_- = -TE_+E_{-+}^{-1}$, so

$$T(E - E_-E_{-+}^{-1}E_-) = 1.$$

So $T^{-1}$ exists and

$$T^{-1}(z) = E(z) - E_+(z)E_{-+}(z)^{-1}E_-(z).$$

Using that $\mathcal{E}\mathcal{P} = 1$, so $E_-R_- = 1$ and $E_-T + E_{-+}R_+ = 0$, we get $T^{-1}$ exists $\implies$ $E_{-+}$ exists.

We get for $z \in N(z_1)$ that $T(z)$ is invertible if and only if $\det E_{-+}(z) \neq 0$. The function $\det E_{-+}(z)$ is holomorphic on $N(z_1)$. So either $\det E_{-+}(z) \equiv 0$, or $\det E_{-+}(z) \neq 0$ in a punctured neighborhood of $z_1$. Let $\Omega_1 = \{z \in \Omega : T(z') \text{ is invertible } \forall z' \neq z \text{ near } z\}$ and $\Omega_2 = \{z \in \Omega : T(z') \text{ is not invertible } \forall z' \neq z \text{ near } z\}$. Then each $\Omega_j$ is open, $\Omega_1 \cup \Omega_2 = \Omega$, and $\Omega_1 \neq \varnothing$ (as $z_0 \in \Omega_1$). Since $\Omega$ is connected, $\Omega_1 = \Omega$ and thus, $\Sigma = \{z \in \Omega : T(z) \text{ is not invertible}\}$ is discrete. ☐

[2] What we lose from this reduction is that if $T(z)$ has some simple dependence of $z$ (e.g. polynomial), $E_{-+}(z)$ may not have a simple dependence. In some contexts, the operator $E_{-+}$ is called the effective Hamiltonian.

**Remark 6.1.** The map $\Omega \setminus \Sigma \to \mathcal{L}(B_2, B_1)$ sending $z \mapsto T(z)^{-1}$ is holomorphic. Consider $T(z)^{-1}$ for $z$ in a punctured neighborhood of $w \in \Sigma$: We have

$$T^{-1}(z) = E(z) - E_+(z)E_{-+}(z)^{-1}E_-(z),$$

where $E, E_+, E_-$ are all holomorphic in a neighborhood of $w$. We have that

$$E_{-+}(z)^{-1} = \frac{\text{holomorphic near } w}{\det E_{-+}(z)},$$

so we have a Laurent expansion

$$E_{-+}(z)^{-1} = \frac{R_{-N_0}}{(z-w)^{N_0}} + \cdots + \frac{R_{-1}}{z-w} + \mathrm{Hol}(z),$$

where $1 \leq N_0 < \infty$ and the $R_j$ are of finite rank. Combining these formulas, we get that $z \mapsto T(z)^{-1}$ has a pole of order $N_0$ at $z = w$:

$$T(z)^{-1} = \frac{A_{-N_0}}{(z-w)^{N_0}} + \cdots + \frac{A_{-1}}{z-w} + Q(z), \qquad Q(z) \text{ holomorphic near } w,$$

where for $1 \leq j \leq N_0$, the $A_{-j} \in \mathcal{L}(B_2, B_1)$ n be expressed in terms of $R_{-N_0}, \dots, R_{-1}$ and are therefore of finite rank.

### 6.2 Application: the residue of the resolvent

Here is an example/special case of the analytic Fredholm theory.

Assume that $B_1 \subseteq B_2$ with continuous inclusion, and let $T(z) = T - z$ for $z \in \Omega$, where $T$ is some operator. Assume that $T(z)$ is Fredholm for each $z$ and that $T(z_0)^{-1}$ exists for some $z_0 \in \Omega$. We get a Laurent expansion for the resolvent $(T-z)^{-1}$ at $w \in \Sigma$:

$$(z-T)^{-1} = \frac{A_{-N_0}}{(z-z_0)^{N_0}} + \cdots + \frac{A_{-1}}{z-w} + Q(z), \qquad Q(z) \text{ holomorphic near } w.$$

for $0 < |z-w| \ll 1$.

**Proposition 6.1.** *The operator $\Pi := A_{-1}$ is a projection*[3] *on $B_2$ which commutes with $T$ (on $B_1$).*

*Proof.* Integrate the Laurent expansion along $\gamma_r = \partial D(w, r)$ for $0 < r \ll 1$. Then

$$\Pi = \frac{1}{2\pi i} \int_{\gamma_r} (z-T)^{-1}\, dz.$$

[3] This is sometimes called the Riesz projection.

We claim that $\Pi^2 = \Pi$: Let $0 < r_1 < r_2 \ll 1$, and write

$$\Pi^2 = \int_{\gamma_{r_2}} \int_{\gamma_{r_1}} (z-T)^{-1}(\widetilde{z}-T)^{-1} \frac{dz}{2\pi i} \frac{d\widetilde{z}}{2\pi i}$$

Using $(\widetilde{z}-T)^{-1} - (z-T)^{-1} = (\widetilde{z}-T)^{-1}(z-\widetilde{z})(z-T)^{-1}$, we have

$$= \int_{\gamma_{r_2}} \int_{\gamma_{r_1}} \frac{1}{\widetilde{z}-z}(z-T)^{-1} \frac{dz}{2\pi i} \frac{d\widetilde{z}}{2\pi i} - \int_{\gamma_{r_2}} \int_{\gamma_{r_1}} \frac{1}{\widetilde{z}-z}(\widetilde{z}-T)^{-1} \frac{dz}{2\pi i} \frac{d\widetilde{z}}{2\pi i}$$

The second term is 0 by applying the Cauchy integral formula on the inner integral.

So we get

$$\Pi^2 = \int_{\gamma_{r_1}} \underbrace{\frac{1}{2\pi i} \int_{\gamma_{r_2}} \frac{1}{\widetilde{z}-z}\, d\widetilde{z}}_{=1}\, (z-T)^{-1} \frac{dz}{2\pi i} = \Pi. \qquad \square$$

**Remark 6.2.** We know that $T(\operatorname{Ran}\Pi) \subseteq \operatorname{Ran}\Pi \subseteq B_1$, where $\operatorname{Ran}\Pi$ is finite dimensional, and let us check that $(T-z_0)|_{\operatorname{Ran}\Pi}$ is nilpotent:

$$\begin{aligned}
(T-z_0)\Pi &= \frac{1}{2\pi i}\int_{\gamma_r} (T-z_0)(z-T)^{-1}\, dz \\
&= \underbrace{\frac{1}{2\pi i}\int_{\gamma_r} (T-z)(z-T)^{-1}\, dz}_{=0} + \frac{1}{2\pi i}\int_{\gamma_r} (z-z_0)(z-T)^{-1}\, dz \\
&= \frac{1}{2\pi i}\int_{\gamma_r} (z-z_0)(z-T)^{-1}\, dz.
\end{aligned}$$

It follows that

$$(T-z_0)^j\Pi = \frac{1}{2\pi i}\int_{\gamma_r} (z-z_0)^j(z-T)^{-1}\, dz.$$

And if $j = N_0$, we get $(T-z_0)^{N_0}\Pi = 0$, as $(z-z_0)^{N_0}(z-T)^{-1}$ is holomorphic.

# 7 Introduction to Unbounded Operators

## 7.1 Motivation from quantum mechanics

In this part of the course, we will discuss spectral theory of self-adjoint operators. We are most interested in unbounded operators, the background of which comes from quantum mechanics.

Classical mechanics: The classical phase space is $\mathbb{R}^{2n} = \mathbb{R}^n_x \times \mathbb{R}^n_\xi$, where $x$ is position and $\xi$ is momentum. Classical observables are, for example, $C^\infty(\mathbb{R}^{2n})$ functions.

**Example 7.1.** The Hamiltonian is

$$p(x,\xi) = |\xi|^2 + V(x),$$

where $V(x)$ is a potential.

In classical dynamics, we have the Hamilton equations

$$\begin{cases} x(t) = p'_\xi(x,\xi) \\ \xi(t) = -p'_x(x,\xi) \end{cases}$$

Quantum mechanics: We have a Hilbert space $H = L^2(\mathbb{R}^n)$. Quantum observables are self-adjoint operators on $H$.

**Example 7.2.** Quantum observables corresponding to $x_j$ and $\xi_j$ are $M_{x_j}$, multiplication by $x_j$, and $D_{x_j} = \frac{1}{i}\partial_{x_j}$. These can not be defined on the whole space, so they will come with their own domains. Associated to $p$ is the **Schrödinger operator**

$$P = -\Delta + V(x).$$

Quantum dynamics is given by the Schrödinger equation

$$i\frac{\partial u}{\partial t} = Pu, \qquad u|_{t=0} = u_0 \in L^2.$$

Formally,

$$u(t) = e^{-itP}u.$$

Interpreting what it means to exponentiate an unbounded operator will be one of the points of our theory.

### 7.2 Unbounded operators

Let $H$ be a complex, separable Hilbert space, let $D \subseteq H$ be a linear subspace, and let $T : D \to H$ be a linear map. Then $D = D(T)$ is the **domain** of $T$. We shall always assume that $T$ is **densely defined**, so that $D(T)$ is dense in $H$. Associated to $T$ is the **graph**[4] of $T$: $G(T) = \{(x, Tx) : x \in D(T)\} \subseteq H \times H$.

**Definition 7.1.** We say that $T$ is **closed** if $G(T)$ is closed subspace of $H \times H$.

**Definition 7.2.** The operator $T$ is **closable** if $\overline{G(T)}$ is the graph of an linear operator $\overline{T} : D(T) \to H$, called the **closure** of $T$.

Note that

$$D(\overline{T}) = \{x \in H : \exists x_j \in D(T) \text{ s.t. } x_j \to x, Tx_j \text{ conv. in } H, \overline{T}x = \lim Tx_j\}.$$

So

$$T \text{ is closed} \iff \text{if } x_n \in D(T), x_n \to x, \text{ and } Tx_n \to y, \text{ then } x \in D(T) \text{ and } Tx = y.$$

On the other hand,

$$\begin{aligned} T \text{ is closable} &\iff G(T) \text{ contains no element of the form } (0, y) \text{ with } y \neq 0 \\ &\iff \text{if } x_n \in D(T),\ x_n \to 0, \text{ and } Tx_n \to y, \text{ then } y = 0. \end{aligned}$$

**Example 7.3.** Let $T = -\Delta$ on $L^2(\mathbb{R}^n)$, with $D(T) = C_0^\infty(\mathbb{R}^n)$. Then $T$ is densely defined and closable: If $\varphi_n \in C_0^\infty$ are s.t. $\varphi \to 0$ in $L^2$ and $\Delta\varphi_n \to \psi \in L^2$, we want $\psi = 0$. For any $f \in C_0^\infty$, $\int \varphi_n f \to 0$, and integrating by parts gives $\int \Delta\varphi_n f = \int \varphi_n \Delta f \to 0$. On the other hand, $\int \Delta\varphi_n f \to \int \psi f$. We get that $\int \psi f = 0$ for all $f \in C_0^\infty$. So $\psi = 0$. In the language of distributions, $\varphi_n \to 0$ in $D'(\mathbb{R}^n)$, so $\Delta\varphi_n \to 0$ in $D'$. So $\psi = 0$.

We claim that $\overline{T} = -\Delta$ with $D(\overline{T}) = H^2(\mathbb{R}^n) = \{u \in L^2 : \partial^\alpha u \in L^2, |\alpha| \leq 2\}$, a **Sobolev space**. Here, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ is a multi-index, $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$, and $|\alpha| = \sum_{j=1}^n \alpha_j$. Here, $\partial^\alpha u \in L^2$ means that there exists some $f_\alpha \in L^2$ such that $\partial^\alpha u = f_\alpha$; that is,

$$(-1)^{|\alpha|} \int u \partial^\alpha \varphi = \int f_\alpha \varphi \qquad \forall \varphi \in C_0^\infty.$$

We have

$$D(\overline{T}) = \{u \in L^2 : \exists \varphi_n \in C_0^\infty \text{ s.t. } \varphi_n \xrightarrow{L^2} u, \Delta\varphi_n \text{ conv. in } L^2\}, \qquad \overline{T}u = \lim(-\Delta\varphi_n).$$

Hence, if $u \in D(\overline{T})$, then $\Delta u = \lim_n \Delta\varphi_n \in L^2$. Then

$$D(\overline{T}) \subseteq \{u \in L^2 : \Delta u \in L^2\} = H^2(\mathbb{R}^n),$$

[4] The idea of thinking about unbounded operators in terms of their graphs goes back to von Neumann.

as taking the Fourier transform, this is

$$D(\overline{T}) \subseteq \{u \in L^2 : (|\xi|^2 + 1)\widehat{u} \in L^2\}.$$

We also have $H^2(\mathbb{R}^n) \subseteq D(\overline{T})$, as $C_0^\infty(\mathbb{R}^n)$ is dense in $H^2(\mathbb{R}^n)$ (the norm on $H^2$ is given by $\|u\|_{H^2} = \sum_{|\alpha|\leq 2} \|\partial^\alpha u\|_{L^2}$). This is the same proof that $C_0^\infty(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$.

We get that $T = -\Delta$ with $D(T) = H^2(\mathbb{R}^n)$ is closed and densely defined.

Next time, we will define what it means for a densely defined operator to be self-adjoint, and we will see that this operator is indeed self-adjoint.

# 8 Realizations of Partial Differential Operators

## 8.1 Maximal and minimal realizations

Last time, we considered the unbounded operator $T = -\Delta$ on $L^2(\mathbb{R}^n)$ with $D(T) = C_0^\infty(\mathbb{R}^n)$. We saw that $\overline{T} = -\Delta$ with domain $D(\overline{T}) = H^2(\mathbb{R}^n) = \{u \in L^2 : \Delta u \in L^2\}$.

**Example 8.1.** Let $\Omega \subseteq \mathbb{R}^n$ be open, and let $P = P(x, D_x)$ be a linear partial differential operator with $C^\infty$ coefficients:

$$P = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha, \qquad a_\alpha \in C^\infty, D_{x_j} = \frac{1}{i}\partial_{x_j}.$$

The operator $P_\Omega$ on $L^2(\Omega)$ with $D(P_\Omega) = C_0^\infty(\Omega)$ is densely defined and closable: if $u_n \in C_0^\infty(\Omega)$ with $u_n \xrightarrow{L^2} 0$ and $Pu_n \xrightarrow{L^2} v$, then $v = 0$. The closure of $P_\Omega$, denoted by $P_{\min}$, is called the **minimal realization** of $P_\Omega$ with domain $D(P_{\min}) = \{u \in L^2 : \exists u_n \in C_0^\infty \text{ s.t. } u_n \to u, Pu_n \text{ conv. in } L^2\}$.

If $u \in D(P_{\min})$, then $Pu \in L^2(\Omega)$, where $Pu$ is defined in the sense of distributions. So $D(P_{\min}) \subseteq \{u \in L^2 : Pu \in L^2\}$. We also introduce the **maximal realization** $P_{\max}$ of $P_\Omega$, given by $D(P_{\max}) = \{u \in L^2 : Pu \in L^2\}$ with $P_{\max}u = Pu$ for all $u \in D(P_{\max})$. We get $P_\Omega \subseteq P_{\min} \subseteq P_{\max}$, meaning $D(P_\Omega) \subseteq D(P_{\min}) \subseteq D(P_{\max})$ and $P_{\max} = P_{\min}$ on $D(P_{\min})$. Both $P_{\min}$ and $P_{\max}$ are closed.

## 8.2 Realizations of order 1 partial differential operators with smooth coefficients

**Proposition 8.1.** *Let $P = \sum_{k=1}^n a_k(x) D_{x_k} + b(x)$ be an operator of order 1 on $\mathbb{R}^n$ with $a_k, b \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $\nabla a_k \in L^\infty$. Then the minimal and the maximal realizations of $P$ agree: $D(P_{\min}) = D(P_{\max})$.*

*Proof.* Let $u \in D(P_{\max})$. We have to show that $u \in D(P_{\min})$; that is, we show there exists a sequence $u_n \in C_0^\infty(\mathbb{R}^n)$ such that $u_n \xrightarrow{L^2} u$ and $Pu_n \xrightarrow{L^2} Pu$. Notice first that if $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi = 1$ near 0 and $\chi_j(x) = \chi(jx)$ for $j = 1, 2, \dots$, then $\chi_j u \xrightarrow{L^2} u$. We may write $P(\chi_j u) = \chi_j Pu + [P, \chi_j]u$. The first term goes to $Pu$ in $L^2$, and

$$[P, \chi_j] = (P \circ \chi_j - \chi_j P)u = \sum_{k=1}^n a_k(x) \frac{1}{j}(D_{x_k}\chi)(x/j) \xrightarrow{L^2} 0.$$

Thus, when proving that $u \in D(P_{\min})$ can be approximated by $C_0^\infty$ functions, we may assume that $u$ has compact support.

Regularize $u$: Let $0 \leq \varphi \in C_0^\infty(\mathbb{R}^n)$ with $\int \varphi = 1$, let $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^n}\varphi(x/\varepsilon)$, and let $J_\varepsilon u = (u * \varphi_\varepsilon)(x) \in C_0^\infty(\mathbb{R}^n)$. Then $J_\varepsilon \xrightarrow{L^2} u$. Compute:

$$P(I_\varepsilon u) = J_\varepsilon Pu + [P, J_\varepsilon]u$$

Since $Pu$ is compactly supported and in $L^2$, the first term goes to $Pu$ in $L^2$. Let's get rid of the $b$ term:

$$[b, J_\varepsilon]u = \underbrace{b(J_\varepsilon u)}_{\to bu} - \underbrace{J_\varepsilon(bu)}_{bu} \xrightarrow{L^2} 0.$$

Since $[P, J_\varepsilon] = \sum [a_k D_{x_k}, J_\varepsilon] + [b, J_\varepsilon]$ it now suffices to show that $[a_k D_{x_k}, J_\varepsilon]u \to 0$ in $L^2$ for all $u \in L^2$. This is Friedrich's lemma. $\square$

**Lemma 8.1** (Friedrich's lemma)**.** *Let* $u \in L^2(\mathbb{R}^n)$ *and* $a_k \in C_0^\infty(\mathbb{R}^n)$*. Then*

$$[a_k D_{x_k}, J_\varepsilon]u \xrightarrow{L^2} 0.$$

*Proof.* Observe first that if $u \in C_0^\infty(\mathbb{R}^n)$, then

$$[a_k D_{x_k}, J_\varepsilon]u = a_k D_{x_k}(J_\varepsilon u) - J_\varepsilon(a_k D_{x_k} u)$$

Since $a_k D_{x_k} u \in C_0^\infty$, the second term goes to $a_j D_{x_k} u$ in $L^2$. The first term also goes to $a_k D_{x_k} u$ in $L^2$. So this goes to 0.

It only remains to show that $\|[a_k D_{x_k}, J_\varepsilon]u\|_{L^2} \leq C\|u\|_{L^2}$ for $0 < \varepsilon \leq 1$ and $u \in C_0^\infty$. Compute

$$\begin{aligned}
W_\varepsilon(x) &= [a_k D_{x_k}, J_\varepsilon]u(x) \\
&= a_k D_{x_k} \int u(y) \frac{1}{\varepsilon^n} \varphi\left(\frac{x-y}{\varepsilon}\right) - \int a_k(x-y) D_{x_k} u(x-y) \frac{1}{\varepsilon^n} \varphi\left(\frac{y}{\varepsilon}\right) dy \\
&= \int a_k(x) u(x - \varepsilon y) \frac{1}{\varepsilon} (D_k \varphi)(y) - \int a_k(x - \varepsilon y) \underbrace{(D_{x_k} u)(x - \varepsilon y)}_{= -1/\varepsilon D_{y_k}(u(x-\varepsilon y))} \varphi(y)\, dy
\end{aligned}$$

Integrate by parts in the second integral.

$$= \int a_k(x) u(x - \varepsilon y) \frac{1}{\varepsilon} (D_{x_k} \varphi)(y) - \int a_k(x - \varepsilon y) u(x - \varepsilon y) D_k \varphi.$$

So we get

$$|W_\varepsilon(x)| \leq C_a |u| * \varphi_\varepsilon.$$

$\square$

# 9 Adjoints of Unbounded Operators

## 9.1 Adjoints

Last time, we showed that if $P = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$ on $L^2(\Omega)$ where $\Omega \subseteq \mathbb{R}^n$ is open and $a_\alpha \in C^\infty(\Omega)$, then we get a **minimal realization**: $P_{\min}$ with $D(P_{\min}) = \{u \in L^2 : \exists \varphi_n \in C_0^\infty(\Omega) : \varphi_n \to u, P\varphi_n \text{ conv.}\}$ given by $P_{\min} u = \lim_{n \to \infty} P\varphi_n$. We also defined the **maximal realization** $P_{\max}$ with $D(P_{\max}) = \{u \in L^2 : Pu \in L^2\}$, where $Pu$ is taken in the sense of distributions. Here, we have $P_{\min} \subseteq P_{\max}$, where both of these are closed operators.

Recall the definition of an adjoint: In a Hilbert space $H$, if $T \in \mathcal{L}(H, H)$, the **adjoint** $T^* \in \mathcal{L}(H, H)$ is defined by $\langle Tx, y \rangle = \langle x, T^* y \rangle$ for all $x, y \in H$. For unbounded operators, we will define this, paying attention to the domains.

**Definition 9.1.** Let $T : D(T) \to H$ be densely defined. We define the **adjoint** $T^*$ by

$$D(T^*) = \{v \in H : \exists f \in H \text{ s.t. } \langle Tu, v \rangle = \langle u, f \rangle \ \forall u \in D(T)\},$$

$$T^* v = f.$$

**Remark 9.1.** The requirement that $T$ is densely defined is crucial to this definition. $D(T)$ is dense, so $f$ is unique if it exists. In particular, $\langle Tu, v \rangle = \langle u, T^* v \rangle$ for all $u \in D(T)$ and $v \in D(T^*)$.

**Remark 9.2.** By the Riesz representation theorem,

$$D(T^*) = \{v \in H : \exists C = C_v > 0 \text{ s.t. } |\langle Tu, v \rangle| \leq C\|u\|, u \in D(T)\}.$$

## 9.2 Examples: adjoints of differential operators

**Example 9.1.** Let $\Omega \subseteq \mathbb{R}^n$ be open, $P = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$ with $a_\alpha \in C^\infty(\Omega)$, where $D = \frac{1}{i}\partial$. Let $P_\Omega = P$ with $D(P_\Omega) = C_0^\infty(\Omega)$. Let's compute $P_\Omega^*$.

First, associated to $P$ is the **formal adjoint** $P^*$ defined by $\langle Pu, v \rangle_{L^2} = \langle u, P^* v \rangle_{L^2}$ for all $u, v \in C_0^\infty(\Omega)$ (such an operator exists for any differential operator). We can calculate the formula using integration by parts:

$$P^* v = \sum_{|\alpha| \leq m} D_x^\alpha (\overline{a_\alpha(x) v}).$$

So $P^*$ is a differential operator of order $m$ with $C^\infty$ coefficients.

To compute the adjoint $P_\Omega^*$, we have

$$\begin{aligned} D(P_\Omega^*) &= \{v \in L^2 : \exists f \in L^2 \text{ s.t. } \langle Pu, v \rangle_{L^2} = \langle u, f \rangle \ \forall u \in C_0^\infty(\Omega)\} \\ &= \{v \in L^2 : P^* v = f \in L^2\}, \end{aligned}$$

where $P^* v$ is taken in the sense of distributions. In other words, $D(P_\Omega^*) = \{v \in L^2 : P^* v \in L^2\} = D(P_{\max}^*)$, the maximal realization of the formal adjoint, and $P_\Omega^* v = P^* v$.

Sometimes, we can give a nice local description of the domain of the adjoint.

**Example 9.2.** Assume that $P = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha$ is **elliptic** in the sense that if $p(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$ for $x \in \Omega, \xi \in \mathbb{R}^n$, then $p(x,\xi) \neq 0$ for all $x \in \Omega, \xi \neq 0$. Then we have

$$\{v \in L^2 : P^* v \in L^2\} \subseteq H^m_{\text{loc}}(\Omega) = \{u \in L^2_{\text{loc}}(\Omega) : \partial^\alpha u \in L^2_{\text{loc}}(\Omega)\ \forall |\alpha| \le m\},$$

a local Sobolev space.

## 9.3 The graph of the adjoint

**Proposition 9.1.** *Let $T : D(T) \to H$ be densely defined. The graph of the adjoint is*

$$G(T^*) = [V(\overline{G(T)})]^\perp,$$

*where $V : H \times H \to H \times H$ sends $(u,v) \mapsto (v,-u)$.*

**Remark 9.3.** Taking the closure is a matter of taste. Since we are taking the orthogonal complement, it does not matter whether or not we close the graph or not, since the result will be closed.

*Proof.* When $u \in D(T)$ and $(v, w^*) \in H \times H$, we have

$$\langle V(u, Tu), (v, w^*) \rangle_{H \times H} = \langle Tu, v \rangle - \langle u, w^* \rangle.$$

The right hand side is 0 for all $u \in D(T)$ if and only if $v \in D(T^*), T^* v = w^*$. This is equivalent to $(v, w^*) \in G(T^*)$. The left hand side is 0 for all $u \in D(T)$ iff $(v, w^*) \in [V(G(T))]$. So $G(T^*) = [V(G(T))]^\perp = [V(\overline{G(T)})]^\perp$. ☐

**Corollary 9.1.** *$T^*$ is closed.*

**Remark 9.4.** If densely defined operators $T_1 \subseteq T_2$ in the sense that $G(T_1) \subseteq G(T_2)$, then $T_2^* \subseteq T_1^*$.

Is $T^*$ densely defined?

**Proposition 9.2.** *$T$ is closable if and only if $D(T^*)$ is dense. In this case, $(T^*)^* = \overline{T}$.*

*Proof.* ($\implies$): Assume there is a nonzero $w \in H$ such that $w \perp D(T^*)$. Then for every $v \in D(T^*)$,

$$\langle (0,w), (T^* v, -v) \rangle_{H \times H} = 0,$$

so $(0,w) \in [V(G(T^*))]^\perp = V(G(T^*)^\perp)$. Recall that $G(T^*) = [V(\overline{G(T)})]^\perp$, so $(0,w) \in V(V(\overline{G(T)}))$. $V^2 = -1$, so $(0,w) \in G(T)$. So $w = 0$, as $T$ is closable.

($\impliedby$): The proof is a similar computation. ☐

# 10 Symmetric and Self-Adjoint Operators

## 10.1 Adjoints of closable operators

If $T : D(T) \to H$ is densely defined, we defined the adjoint $T^*$ with $G(T^*) = [\overline{V(G(T))}]^\perp$, where $V(u,v) = (v, -u)$. Let's finish a proof we started last time.

**Proposition 10.1.** *$T$ is closable if and only if $T^*$ is densely defined.*

*Proof.* ( $\implies$ ): We did this last time.

( $\impliedby$ ): If $D(T^*)$ is dense, then $(T^*)^*$ is a closed operator such that

$$F(T^{**}) = [V(G(T^*))]^\perp = V(G(T^*)^\perp) = V(V(\overline{G(T)})) = \overline{G(T)},$$

where we have used $V^2 = -1$. So $T$ is closable, and $T^{**} = \overline{T}$. □

## 10.2 Symmetric and self-adjoint operators

**Definition 10.1.** Let $S : D(S) \to H$ be densely defined. We say that $S$ is **symmetric** if $\langle Sx, y\rangle = \langle x, Sy\rangle$ for all $x, y \in D(S)$.

**Example 10.1.** $S = -\Delta$ on $L^2(\mathbb{R}^n)$ with $D(S) = C_0^\infty(\mathbb{R}^n)$ is symmetric. However, we will see that this operator is not self-adjoint.

$S$ is symmetric if and only if $S \subseteq S^*$.

**Proposition 10.2.** *If $S$ is symmetric, then $S$ is closable and $\overline{S}$ is symmetric.*

*Proof.* If $u_n \in D(S)$ with $u_n \to 0$ and $Su_n \to \ell$, then $\langle u_n, Sv\rangle = \langle Su_n, v\rangle \to \langle \ell, v\rangle$. On the other hand $\langle u_n, Sv\rangle \to 0$, for all $v \in D(S)$. So $\ell = 0$, and $S$ is closable.

If $S \subseteq S^*$, where $S^*$ is densely defined, then $\overline{S} = S^{**} \subseteq S^* = \overline{S}^*$. So $\overline{S}$ is symmetric. □

Given a symmetric operator $S$, we have two natural closed extensions: $\overline{S}$ and $S^*$.

**Definition 10.2.** A linear, densely defined operator $T : D(T) \to H$ is called **self-adjoint** if $T = T^*$.

Note that this means that $D(T) = D(T^*)$. Any self-adjoint operator is closed, since adjoints are closed. We have

$$\begin{aligned} T \text{ is self-adjoint} &\iff T \text{ is symmetric and } D(T) = D(T^*) \\ &\iff \langle Tx, y\rangle = \langle x, Ty\rangle \quad \forall x, y \in D(T), \text{ and} \\ &\qquad \text{if } (x,y) \in H \times H \text{ with } \langle Tz, x\rangle = \langle z, y\rangle \ \forall z \in D(T) \implies x \in D(T). \end{aligned}$$

**Proposition 10.3.** *Let $S$ be closed and symmetric. Then $S^*$ is symmetric, so $S$ is self-adjoint.*

*Proof.* $S^*$ is symmetric, so $S^* \subseteq S^{**} = S$, as $S$ is closed. Also, $S \subseteq S^*$, so $S$ is self-adjoint. $\square$

**Example 10.2.** Let $H = L^2(\mathbb{R}^n)$, and let $m : \mathbb{R}^n \to \mathbb{R}$ be Lebesgue measurable. Let $D(A) = \{f \in L^2 : mf \in L^2\}$ and $Af = mf$ for all $f \in D(A)$. We claim that $A$ is self-adjoint.

Check first that $D(A)$ is dense in $L^2$: For any $f \in L^2$, $\frac{f}{1+|m|} \in L^2$, as well. So if $g \in L^2$ with $g \perp D(A)$,

$$\int g \frac{f}{1+|m|}\, dx = 0,$$

which means that $\frac{g}{1+|m|} = 0$, giving $g = 0$.

$A$ is symmetric, as $m$ is real. Now let $(g, h) \in L^2 \times L^2$ be such that $\langle Af, g\rangle = \langle f, h\rangle$ for all $f \in D(H)$. Then for all $f \in L^2$,

$$\int \frac{mf}{1+|m|}\overline{g} = \int \frac{f}{1+|m|}\overline{h},$$

so

$$\int \left(\frac{m}{1+|m|}\overline{g} - \frac{\overline{h}}{1+|m|}\right) f = 0$$

for all $f \in L^2$. So $mg = h$, which gives $g \in D(A)$.

**Example 10.3.** Let $T = -\Delta$ on $L^2(\mathbb{R}^n)$ with $D(T) = H^2(\mathbb{R}^n) = \{u \in L^2 : \Delta u \in L^2\} = \{u \in L^2 : \partial^\alpha u \in L^2\ \forall |\alpha| \le 2\}$. Then $T$ is self-adjoint.

$T$ is symmetric: $\langle -\Delta u, v\rangle_{L^2} = \langle u, -\Delta v\rangle_{L^2}$ for all $u, v \in H^2$. This is true for $u, v \in C_0^\infty(\mathbb{R}^n)$, which is dense in $H^2(\mathbb{R}^n)$. Alternatively we could prove this by taking the Fourier transform, where $T$ acts as a multiplication operator.

Let $(g, h) \in L^2 \times L^2$ be such that $\langle -\Delta u, g\rangle_{L^2} = \langle u, h\rangle$ for all $g, h \in H^2$. In particular, if $u \in C_0^\infty$, we get $-\Delta g = h \in L^2$ (taken in the weak sense). Then $g \in H^2$, and $Tg = -\Delta g = h$.

## 10.3 von Neumann's extension theory for symmetric operators

Let $S : D(S) \to H$ be a closed, symmetric (densely defined) operator. Can $S$ be extended to a self-adjoint operator? If $S \subseteq T = T^*$, then $T^* \subseteq S^*$, so we have an operator between $S$ and $S^*$ in general. If $S$ is symmetric, these are the same.

**Proposition 10.4.** *Let $S$ be closed and symmetric. Then for any $z \in \mathbb{C} \setminus \mathbb{R}$, $S - z1 : D(S) \to H$ is injective, has closed range, and $\|(S - z)u\| \ge |\operatorname{Im} z|\|u\|$ for $u \in D(S)$.*

*Proof.* Write $z = x + iy$. Then

$$\|(S - z)u\|^2 = \langle (S - x)u + iyu, (S - x)u + iyu\rangle$$

$$
\begin{aligned}
&= \|(S-x)u\|^2 + y^2\|u\|^2 \\
&\geq y^2\|u\|^2,
\end{aligned}
$$

so $S - z$ is injective.

$\mathrm{Im}(S-z)$ is closed: If $y \in \overline{\mathrm{Im}(S-z)}$, there exist $x_n \in D(S)$ such that $(S-z)x_n \to y$. By this inequality, $x_n \to x \in H$. Since $(S-z)$ is closed, $x \in D(S)$. $\square$

Here is the idea due to von Neumann: Study the **Cayley transform** of $S$, $T = (S+i)(S-i)^{-1}$.

# 11 The Cayley Transform of Symmetric Operators

## 11.1 The Cayley transform

Let $S : D(S) \to H$ be closed, symmetric, and densely defined. We have shown that $S \pm i$ are injective iff $\operatorname{Im}(S \pm i)$ is closed, and $\|(S+i)x\|^2 = \|(S-i)x\|^2$ $(= \|Sx\|^2 + \|x\|^2)$.

**Definition 11.1.** The Cayley transform $T$ of $S$ is the operator $T = (S+i)(S-i)^{-1} : \operatorname{Im}(S-i) \to \operatorname{Im}(S+i)$.

The above norm calculation shows that $T$ is an isometric bijection.

**Proposition 11.1.** *$T-1$ is injective,* $\operatorname{Im}(T-1) = D(S)$*, and* $S = i(T+1)(T-1)^{-1} : D(S) \to H$.

*Proof.* If $y \in \operatorname{Im}(S-i)$ with $y = (S-i)x$, then

$$(T-1)y = (S+i)x - (S-i)x = 2ix.$$

We get $T-1$ is injective and $\operatorname{Im}(T-1) = D(S)$. Similarly,

$$(T+1)y = (S+i)x + (S-i)x = 2Sx,$$

so

$$2S\frac{1}{2i}(T-1)y = (T+1)y.$$

Then

$$S = i(T+1)(T-1)^{-1}.$$

□

Conversely, let $H_1, H_2 \subseteq H$ be closed subspaces, and let $T : H_1 \to H_2$ be a unitary map be such that $\operatorname{Im}(T-1)$ is dense in $H$. We claim that $T-1$ is injective: If $(T-1)y = 0$ for $y \in H_1$, then for $z \in H_1$,

$$\langle y, (T-1)z \rangle = \langle y, Tz \rangle - \langle y, z \rangle = \langle Ty, Tz \rangle - \langle y, z \rangle = 0.$$

Define $S : D(S) = \operatorname{Im}(T-1) \to H$ by $S = i(T+1)(T-1)^{-1}$. We claim that $S$ is symmetric. For $x = (T-1)y \in D(S)$,

$$\begin{aligned} \langle Sx, x \rangle &= i \langle (T+1)y, (T-1)y \rangle \\ &= i(\|Ty\|^2 - \langle Ty, y \rangle + \langle y, Ty \rangle - \|y\|^2) \\ &= i(-\langle Ty, y \rangle + \langle y, Ty \rangle) \in \mathbb{R}. \end{aligned}$$

We get $\langle Sx, x \rangle = \langle x, Sx \rangle$ for all $x \in D(S)$. Polarize this identity (i.e. $x = y+z, y+iz$) to get that $S$ is symmetric.

We claim that $S$ is closed. If $(x,z) \in \overline{G(S)}$, there is a sequence $y_n \in H_1$ such that $(T-1)y_n \to x$ and $i(T+1)y_n \to z$. So $y_n \to y \in H_1$. $Ty_n \to Ty$, so $(T-1)y_n \to (T-1)y = x \in D(S)$. Then

$$i(T+1)y_n \to i(T+1)y = i(T+1)(T-1)^{-1}x = Sx = z.$$

Finally, let $T_1$ be the Cayley transform of $S$, $T_1 : \operatorname{Im}(S-i) \to \operatorname{Im}(S+i)$ with $T_1 = (S+i)(S-i)^{-1}$. If $y \in D(S) = \operatorname{Im}(T-1)$ with $y = (T-1)x$ $(x \in H_1)$, then

$$(S-i)y = (S-i)(T-1)x = i(T+1)x - i(T-1)x = 2ix.$$

So $D(T_1) = H_1 = D(T)$. Now we check

$$T_1 x = \frac{1}{2i} T_1 (S-i)y = \frac{1}{2i}(S+i)y = \frac{1}{2i}(\underbrace{S(T-1)x}_{i(T+1)x} + i(T-1)x) = Tx.$$

So the Cayley transform of $S$ is $T$.

We summarize the results in a proposition.

**Proposition 11.2.** *Let $S$ be closed, symmetric, and densely defined. Then the Cayley transform $T : \operatorname{Im}(S-i) \to \operatorname{Im}(S+i)$ sending $(S-i)x \mapsto (S+i)x$ is unitary, $\operatorname{Im}(T-1) = D(S)$, $T-1$ is injective, and $S = i(T+1)(T-1)^{-1}$. Conversely, if $H_1, H_2$ are closed subspaces of $H$, $T : H_1 \to H_2$ is unitary, and $\operatorname{Im}(T-1)$ is dense, then $T$ is the Cayley transform of a unique symmetric, closed, densely defined operator $S$.*

## 11.2 Deficiency subspaces

Now we are ready to check whether a closed, symmetric, and densely defined operator is self-adjoint.

**Definition 11.2.** Let $S$ be closed, symmetric, and densely defined. The **deficiency subspaces** associated to $S$ are $D_\pm := (\operatorname{Im}(S \pm i))^\perp = \ker(S^* \mp i)$. The **deficiency indices** are $n_\pm = \dim D_\pm$ (Hilbert space dimension).

The deficiency indices measure the extent to which $S$ may fail to be self-adjoint.

We know $D_\pm \subseteq D(S)$. Introduce also

$$D(S^*)|_{D_\pm} =: \widehat{D}_\pm = \{(x, S^*x) : x \in D_\pm\} = \{\langle x, \pm ix\rangle : x \in D_\pm\}.$$

**Theorem 11.1.** *Let $S$ be closed, symmetric, and densely defined. Then*

$$G(S^*) = G(S) \oplus \widehat{D}_+ \oplus \widehat{D}_-,$$

*where the direct sum is orthogonal.*

*Proof.* Check first that $G(S) \perp \widehat{D}_{\pm}$ (we check $+$): if $x \in D(S)$ and $y_+ \in D_+$

$$\langle (x, Sx), (y_+, iy_+) \rangle = \langle x, y_+ \rangle + \langle Sx, iy_+ \rangle = \langle x, y_+ \rangle + \underbrace{\langle x, iS^* y_+ \rangle}_{=\langle x, iiy_+ \rangle} = 0.$$

Also, $\widehat{D}_+ \perp \widehat{D}_-$:

$$\langle (y_+, iy_+), (y_-, -iy_-) \rangle = \langle y_+, y_- \rangle + \langle iy_+, (1/i) y_- \rangle = 0.$$

By orthogonality, $G(S) \oplus \widehat{D}_+ \oplus \widehat{D}_-$ is a closed subspace of $G(S^*)$. It remains to show that if $(y, S^* y)$ is orthogonal to $G(S), \widehat{D}_{\pm}$, then $y = 0$. We will show this next time. $\square$

# 12 Extending Symmetric Operators to Self-Adjoint Operators

## 12.1 Graph of the adjoint of a symmetric operator

Suppose we have a closed, symmetric, densely defined operator $S : D(S) \to H$. We introduced the **deficiency subspaces** $D_\pm = (\mathrm{Im}(S \pm i))^\perp = \ker(S^* \mp i)$ and the graphs $\widehat{D}_\pm = G(S^*)|_{D_\pm}$.

Last time, were proving the following theorem.

**Theorem 12.1.** *Let $S$ be closed, symmetric, and densely defined. Then*

$$G(S^*) = G(S) \oplus \widehat{D}_+ \oplus \widehat{D}_-,$$

*where the direct sum is orthogonal.*

*Proof.* It remains to show that if $(y, S^*y) \perp G(S), \widehat{D}_\pm$, then $y = 0$.

First, if $(y, S^*y) \perp G(S)$, then $\langle (y, S^*y), (x, Sx) \rangle = 0$ for all $x \in D(S)$. Then $\langle Sx, S^*y \rangle + \langle x, y \rangle = 0$ for all $x \in D(S)$. So $S^*y \in D(S^*)$ and $S^*(S^*y) = -y$. So $((S^*)^2 + 1)y = 0$. We get $(S^* - i)(S^* + i)y = 0$, so $(S^* + i)y \in D_+$.

If $(y, S^*y) \perp \widehat{D}_+$, then $\langle (y, S^*y), (x, ix) \rangle = 0$ for all $x \in D_+$. We get $\langle y, x \rangle + \langle S^*y, ix \rangle = 0$, so $-i \langle (S^* + i)y, x \rangle = 0$ for all $x \in D_+$. So $(S^* + i)y = 0$.

Similarly, $(S^* - i)y \in D_-$ (changing the order in the factorization). Then $(y, S^*y) \perp \widehat{D}_-$, so $(S^* - i)y = 0$. So we get $y = 0$. □

## 12.2 Conditions for extending symmetric operators

**Corollary 12.1.** *A symmetric, closed operator $S : D(S) \to H$ is self-adjoint if and only if the deficiency indices $n_+ = n_- = 0$, or equivalently,* $\mathrm{Im}(S \pm i) = H$. *Equivalently, the Cayley transform of $S$ is unitary : $H \to H$.*

In general, we have the following:

**Corollary 12.2.** *A symmetric, closed operator $S : D(S) \to H$ has a self-adjoint extension if and only if the Cayley transform $T$ can be extended to a unitary map: $H \to H$.*

*Proof.* This follows from our correspondence between symmetric operators and their Cayley transforms. □

Using the full strength of this result we have proven, we get the original result of von Neumann's extension theory.

**Theorem 12.2** (von Neumann)**.** *A closed, densely defined, symmetric operator $S : H \to H$ has a self-adjoint extension if and only if the deficiency indices are equal.*

*Proof.* Assume first that $T$ can be extended to a unitary map $U : H \to H$ (so $U|_{\mathrm{Im}(S-i)} = T$). Write $H = D(T) \oplus D_-$ and $H = \mathrm{Im}(T) \oplus D_+$ (orthogonal decompositions). It follows that $U|_{D^-} : D_- \to D_+$ is a bijection, so the deficiency indices are equal: $n_- = n_+$.

Conversely, assume that $n_- = n_+$. Let $(e_j^+)_{j\in J}, (e_j^-)_{j\in J}$ be orthonormal bases for $D_+$ and $D_-$, respectively. Let $T_1 : D_- \to D_+$ take $\sum_{j\in J} x_j e_j^- \mapsto \sum_{j\in J} x_j e_j^+$. $T_1$ is unitary, so the map $U : H \to H$ sending $(y + z) \mapsto Ty + T_1 z$ (where $y \in D(T), z \in D_-$) is a unitary extension of $T$. $\square$

**Remark 12.1.** We say that closed, symmetric operator $S$ is **maximal** if it has no strict symmetric extension. If $S$ is self-adjoint, it is maximal. In general, $S$ is maximal if and only if at least one of the deficiency indices equals 0.

### 12.3 Example: Extending the Schrödinger operator

**Example 12.1.** The Schrödinger operator: $H = L^2(\mathbb{R}^n)$, and $P = -\Delta + V(x)$, where $V \in L^2_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R})$. Equipped with the domain $C_0^\infty(\mathbb{R}^n)$, $P$ becomes symmetric and densely defined.

We claim that $P$ has a self-adjoint extension. We have to check that $n_+ = \dim\ker(P^* - i) = \dim\ker(P^* + i) = n_-$. Here, $D(P^*) = \{u \in L^2 : Pu \in L^2\}$, where $Pu$ is taken in the sense of distributions; $Vu \in L^1_{\mathrm{loc}}$, so it makes sense as a distribution. The complex conjugation map $\Gamma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ sending $u \mapsto \overline{u}$ satisfies: $\Gamma(D(P^*)) \subseteq D(P^*)$ and $[\Gamma, P^*] = 0$. Since $\Gamma : D_- \to D_+$ is a bijection, the deficiency indices are equal. (Here, we use that $P$ is real.)

# 13 Examples of Self-Adjoint Extensions

## 13.1 Self-adjoint extensions of differential operators

Let $S : D(S) \to H$ be symmetric, closed, and densely defined. Last time, we made the observation that $S$ is self-adjoint $\iff \operatorname{Im}(S \pm i) = H \iff \ker(S^* \mp i) = \{0\}$. We also saw that $S$ has a self-adjoint extension $\iff \dim \operatorname{Im}(S+i)^\perp = \dim \operatorname{Im}(S-i)$.

**Example 13.1.** Let $H = L^2(\mathbb{R}^n)$, and let $P = P(D)$ be a linear, differential operator with constant, real coefficients:

$$P = \sum_{|\alpha| \le m} a_\alpha D^\alpha, \qquad a_\alpha \in \mathbb{R}, D = \frac{1}{i}\partial.$$

Let $P_{\min}$ be the minimal realization of $P$: $P_{\min} = \overline{P|_{C_0^\infty}}$. Then $P_{\min}$ is closed, densely defined, and symmetric: if $u, v \in C_0^\infty$,

$$\langle Pu, v \rangle_{L^2} = \int Pu\overline{v}\,dx = \sum_{|\alpha| \le m} \int a_\alpha D^\alpha u \overline{v}\,dx = \langle u, Pv \rangle_{L^2}.$$

We claim that $P_{\min}$ is self-adjoint. Check that $\ker(P_{\min}^* \pm i) = \{0\}$: Here, $D(P_{\min}^*) = \{u \in L^2 : Pu \in L^2\}$. If $u \in D(P_{\min})$, then we get a differential equation:

$$(P_{\min}^* \pm i)u = 0 \iff (P(D) \pm i)u = 0$$

Take the Fourier transform:

$$\mathcal{F}[(P(D) \pm i)u] = 0 \iff \left( \sum_{|\alpha| \le m} a_\alpha \xi^\alpha \pm i \right) \widehat{u}(\xi) = 0.$$

Then $\widehat{u} = 0$, so $u = 0$.

Since $P_{\max} = P_{\min}^*$, we get that $P_{\max} = P_{\min}$. So $P$ has only one realization, which is self-adjoint. That is, it only has one self-adjoint extension.

**Example 13.2.** Let $H = L^2((0,\infty))$, and let $P(D) = D = \frac{1}{i}\frac{d}{dx}$. Let $P_{\min} = \overline{P|_{C_0^\infty}}$. Compute the deficiency indices: $(P(D) \pm i)i = 0$ for $u \in L^2((0,\infty))$, so

$$\left( \frac{1}{i}\frac{d}{dx} - i \right) u = 0 \iff u' + u = 0 \iff u(x) = Ce^{-x} \in L^2.$$

So $n_+ = 1$. For the $+$ case, we have

$$\left( \frac{1}{i}\frac{d}{dx} + i \right) u = 0 \iff u' - u = 0 \iff u(x) = Ce^{x}.$$

But such a $u \notin L^2((0,\infty))$, so $n_- = 0$.

Thus, $P_{\min}$ is maximal, symmetric, and has no self-adjoint extensions.

**Remark 13.1.** We have omitted the argument that these differential equations have no nonclassical solutions. We have

$$u' + u = 0 \iff (e^x u)' = 0,$$

where this derivative is in the distributional sense. We use the fact that if $u \in D'(\mathbb{R})$ with $u' = 0$, then $u$ is constant.

**Remark 13.2.** In this example, $D(P_{\min}^*) = \{u \in L^2 : Pu \in L^2\} = H^1((0, \infty))$.

### 13.2 Essentially self-adjoint operators

**Definition 13.1.** Let $S : D(S) \to H$ be symmetric and densely defined. We say that $S$ is **essentially self-adjoint** if $\overline{S}$ is self-adjoint.

Here is an example.

**Theorem 13.1** (Essential self-adjointness of the Schrödinger operator with a semibounded potential)**.** *Let $P = P(x, D) = -\Delta + q(x)$, where $q \in C(\mathbb{R}^n; \mathbb{R})$. Let $P_0$ be the minimal realization of $P$: $P_0 = \overline{P|_{C_0^\infty}}$, which is closed, symmetric and densely defined. Assume that $q \geq -C$ on $\mathbb{R}^n$. Then $P_0$ is self-adjoint (i.e. $P(x, D)$ is essentially self-adjoint).*

**Remark 13.3.** $-\Delta \geq 0$: If $u \in C_0^\infty$, $\langle -\Delta u, u\rangle = \int -\Delta u\overline{u} = \int |\nabla u|^2 \geq 0$. We cannot let the operator tend to $-\infty$ unchecked, which is why we need this semiboundedness condition. This condition can be relaxed, but there needs to be some condition.

If $q$ were actually bounded, this theorem is easier to prove. One can prove that a self adjoint operator plus a bounded self-adjoint operator is still self-adjoint (and with the same domain).

*Proof.* $D(P_0^*) = \{u \in L^2 : Pu = (-\Delta + q)u \in L^2\}$, and $P_0^* u = Pu$ for $u \in D(P_0^*)$. We shall show that $P_0^*$ is symmetric; that is, $\langle u, P_0^* u\rangle_{L^2} \in \mathbb{R}$ for all $u \in D(P_0^*)$. First, if $u \in D(P_0^*)$, then $\Delta u \in L^2_{\text{loc}}$. So $u \in H^2_{\text{loc}} = \{u \in L^2_{\text{loc}} : \partial^\alpha u \in L^2_{\text{loc}}\ \forall |\alpha| \leq 2\}$. In particular, $\nabla u \in L^2_{\text{loc}}$.

We claim that if $u \in D(P_0^*)$, then $\nabla u \in L^2(\mathbb{R}^n)$. We may assume that $u \in D(P_0^*)$ is real (by considering real and imaginary parts separately). Consider

$$\int \psi_t(x) i Pu\, dx = \int \psi_t(x) u(-\Delta + q)u\, dx,$$

where $\psi_t(x) = \psi(tx)$, $0 \leq \psi \in C_0^\infty(\mathbb{R}^n)$ is a cutoff which is 1 near 0. The idea is that once we introduce this cutoff, we can integrate by parts. We will get something like $\int \psi_t |\nabla u|^2$ and will try to control this uniformly in $t$ to use Fatou's lemma. □

We will finish the proof next time.

# 14 Essential Self-Adjointness of Schrödinger Operators and Perturbations of Self-Adjoint Operators

## 14.1 Essential self-adjointness of Schrödinger operators

Last time, we were proving the following theorem.

**Theorem 14.1** (Essential self-adjointness of the Schrödinger operator with a semibounded potential)**.** *Let $P = P(x,D) = -\Delta + q(x)$, where $q \in C(\mathbb{R}^n;\mathbb{R})$. Let $P_0$ be the minimal realization of $P$: $P_0 = \overline{P|_{C_0^\infty}}$, which is closed, symmetric and densely defined. Assume that $q \geq -C$ on $\mathbb{R}^n$. Then $P_0$ is self-adjoint (i.e. $P(x,D)$ is essentially self-adjoint).*

*Proof.* Let $P_0 = \overline{P_{C_0^\infty}}$, so $D(P_0^*) = \{u \in L^2 : Pu \in L^2\} \subseteq H^2_{\text{loc}}$. We shall show that $P_0^*$ is symmetric, which is equivalent to $\langle u, P_0^* u\rangle_{L^2} \in \mathbb{R}$ for all $u \in D(P_0^*)$.

We claim that for every $u \in D(P_0^*)$, $\nabla u \in L^2(\mathbb{R}^n)$. It suffices to show this claim when $u$ is real (by splitting up real and imaginary parts). Consider

$$\int \psi_t^2 u P_0^* u\, dx = \int \psi_t^2 u P u\, dx,$$

where $\psi_t(x) = \psi(tx)$ for $t > 0$, $0 \leq \psi \in C_0^\infty$, and $\psi(x) = 1$ in $|x| \leq 1$. Write

$$\begin{aligned} \int \psi_t^2(x) u P u\, dx &= \int \psi_t^2 u(-\Delta + q) u\, dx \\ &= \int \psi_t^2 u(-\Delta u) + \int \psi_t^2 q u^2 \end{aligned}$$

Integrating by parts in the first integral (we can integrate $u$ by parts by regularizing it, but we omit that argument),

$$= \int \nabla(\psi_t^2 u) \cdot \nabla u + \int \psi_t^2 q u^2$$

We get

$$\underbrace{\int \psi_t^2(x) u P u\, dx}_{\leq \|u\|_{L^2} \|Pu\|_{L^2}} = \int \psi_t^2 |\nabla u|^2 + \int 2\psi_t \nabla \psi_t \cdot \nabla u + \underbrace{\int q \psi_t u^2}_{-C \int \psi_t^2 u^2 \geq -C\|u\|^2}.$$

Let $I(T) = \int \psi_t |\nabla u|^2$. We get that

$$I(T) \leq O(1) + 2 \underbrace{\int |\psi_t \nabla u \cdot u \nabla \psi_t|}_{\leq C I(t)^{1/2} \|u\|_{L^2}}$$

$$\leq O(1) + CI(t)^{1/2}.$$

This implies that $I(t) \leq O(1)$ because $CI(t)^{1/2} \leq C^2 \frac{1}{\varepsilon} + \varepsilon I(T)$ for all $\varepsilon > 0$ by the AM-GM inequality. The claim follows by Fatou's lemma.

Let $u \in D(P_0^*)$ be complex-valued. Then

$$\int \psi_t^2 u \overline{Pu} = \underbrace{\int \psi_t^2 |\nabla u|^2}_{\in \mathbb{R}} + \underbrace{2 \int \psi_t u \nabla \psi_t \cdot \overline{\nabla u}}_{\leq 2\|\nabla \psi_t\|_{L^\infty} \|u\|_{L^2} \|\nabla u\|_{L^2} \to 0} + \underbrace{\int q \psi_t^2 |u|^2}_{\in \mathbb{R}}$$

so the imaginary part of this goes to 0 as $t \to \infty$. Also, $\int \psi_t^2 u \overline{Pu} \to \int u \overline{Pu}$ as $t \to \infty$, so $\int u \overline{Pu} = \langle u, P_0^* u \rangle_{L^2} \in \mathbb{R}$. $\square$

**Example 14.1.** The quantum harmonic oscillator is the case of $q(x) = |x|^2$, so $P = -\Delta + |x|^2$ is essentially self-adjoint on $C_0^\infty$. One can show that the domain is $D(P_0) = \{u \in L^2 : x^\alpha \partial^\beta \in L^2, |\alpha + \beta| \leq 2\}$.

**Remark 14.1.** If $S$ is essentially self-adjoint, then $\overline{S} = (\overline{S})^* = S^*$. So the closure is the adjoint. In particular, there is only 1 realization.

### 14.2 Perturbations of self-adjoint operators

Let $A : D(A) \to H$. Then $A$ is closed if and only if $D(A)$ is a Banach space with respect to the **graph norm**: $\|u\|_{D(A)} := \|u\| + \|Au\|$.

**Definition 14.1.** Let $A, B$ be linear operators on $H$. We say that $B$ is $A$**-bounded** (or **relatively bounded with respect to** $A$) if $D(B) \supseteq D(A)$ and if there are constants $a, b \geq 0$ such that

$$\|Bu\| \leq a\|Au\| + b\|u\|, \qquad \forall u \in D(A).$$

The infimum of all such constants $a$ is the **relative bound** of $B$ with respect to $A$.

**Proposition 14.1.** *Let $A$ be closed, and let $B$ be $A$-bounded with a relative bound $< 1$. Then $A + B$ is closed on $D(A)$.*

*Proof.* We have:

$$\|Bu\| \leq a\|Au\| + b\|u\|, \qquad \forall u \in D(A)$$

with $a < 1$. Check that the norms $u \mapsto \|u\| + \|Au\|$ and $u \mapsto \|u\| + \|(A + B)u\|$ are equivalent on $D(A)$. So $A + B$ is closed. $\square$

**Theorem 14.2** (Kato-Rellich[5])**.** *Let $A$ be self-adjoint, and let $B$ be symmetric and $A$-bounded with relative bound $< 1$. Then $A + B$ is self-adjoint on $D(A)$.*

[5] Kato and Rellich both proved this result around the same time, independently of each other.

*Proof.* $A+B$ is closed, symmetric, and densely defined on $D(A)$. So we only need to show that the deficiency indices are 0: that is, we want $\mathrm{Im}(A+B\pm i)=H$. In fact, we will show that there exists some $\lambda\in\mathbb{R}\setminus\{0\}$ such that $\mathrm{Im}(A+B\pm i\lambda)=H$. $\square$

We will prove this next time.

# 15 The Kato-Rellich Theorem

## 15.1 The Kato-Rellich theorem

Last time, we were in the middle of proving the Kato-Rellich theorem.

**Theorem 15.1** (Kato-Rellich)**.** *Let $A$ be self-adjoint, and let $B$ be symmetric and $A$-bounded with relative bound $< 1$. Then $A + B$ is self-adjoint on $D(A)$.*

*Proof.* $A + B$ is closed, symmetric, and densely defined on $D(A)$. So we only need to show that the deficiency indices are 0: that is, we want $\operatorname{Im}(A + B \pm i) = H$. In fact, we will show that there exists some $\lambda \in \mathbb{R} \setminus \{0\}$ such that $\operatorname{Im}(A + B \pm i\lambda) = H$.

As $A$ is self-adjoint, this is true when $B = 0$. We have

$$\|(A + i\lambda)u\|^2 = \|Au\|^2 + \lambda^2\|u\|^2 \qquad \forall u \in D(A).$$

So we know that $A + i\lambda : D(A) \to H$ is bijective, and

$$\|v\|^2 = \|A(A + i\lambda)^{-1}v\|^2 + \lambda^2\|(A + i\lambda)^{-1}v\|^2 \qquad \forall v \in H.$$

So $(A + i\lambda)^{-1}, A(A + i\lambda)^{-1} \in \mathcal{L}(H, H)$ with

$$\|(A + i\lambda)^{-1}\| \leq \frac{1}{|\lambda|}, \qquad \|A(A + i\lambda)^{-1}\| \leq 1.$$

Next by the $A$-boundedness of $B$, there exists some $0 \leq a < 1$ such that for any $u \in H$,

$$\begin{aligned}
\|B(A + i\lambda)^{-1}u\| &\leq a\|A(A + i\lambda)^{-1}u\| + b\|(A + i\lambda)^{-1}u\| \\
&\leq a\|u\| + \frac{b}{|\lambda|}\|u\| \\
&= \left(a + \frac{b}{|\lambda|}\right)\|u\|
\end{aligned}$$

Pick $\lambda$ large enough to get

$$= \left(\frac{1 + a}{2}\right)\|u\|.$$

Thus, the operator $1 + B(A + i\lambda)^{-1}$ is invertible in $\mathcal{L}(H, H)$. We get that

$$A + B + i\lambda = (1 + B(A + i\lambda)^{-1})(A + i\lambda) : D(A) \to H$$

is bijective. So $A + B$ is self-adjoint on $D(A)$. □

Here is an application:

**Example 15.1** (Schrödinger operator with a Coulomb potential)**.** Let $H = L^2(\mathbb{R}^3)$, and let $P_0 = -\Delta$ (self-adjoint with $D(P_0) = H^2(\mathbb{R}^3)$). Our potential is $V(x) = \frac{\gamma}{|x|}$ with $\gamma \in \mathbb{R}$.

We claim that $P = P_0 + V$ is self-adjoint on $L^2$ with domain $D(P) = H^2$. We may assume that $|\gamma|$ is small, for we can change scales: Introduce $U_\lambda : L^2 \to L^2$ which acts as $(U_\lambda f)(x) = \lambda^{-n/2} f(x/\lambda)$. Then

$$U_\lambda^{-1}(-\Delta + V)U_\lambda = -\frac{1}{\lambda^2}\Delta + V(\lambda\,\cdot) = -\frac{1}{\lambda^2}\Delta + \frac{\gamma}{\lambda|x|} = \frac{1}{\lambda^2}\left(-\Delta + \frac{\lambda\gamma}{|x|}\right).$$

So we don't need to worry so much about the relative bound in the Kato-Rellich theorem.

We shall show that

$$\|Vu\|^2 = \int \frac{|u(x)|^2}{|x|^2}\,dx \le C(\|P_0 u\| + \|u\|)^2 \qquad \forall u \in D(P_0).$$

Let $\chi \in C_0^\infty(\mathbb{R}^3)$ be

$$\chi = \begin{cases} 1 & |x| < 1 \\ 0 & |x| > 2. \end{cases}$$

Then, letting $E(x) = -1/(4\pi|x|)$ be the Newtonian potential in $\mathbb{R}^3$ (so $\Delta E = \delta_0$),

$$\chi u = \delta_0 * \chi u = \Delta E * \chi u = \underbrace{E}_{\in L^2_{\text{loc}}} * \underbrace{\Delta(\chi u)}_{\in L^2_{\text{compact}}}.$$

So $\chi u \in L^\infty$ is continuous, and

$$|\chi u(x)| = \left|\int E(y)\Delta(\chi u)(x-y)\,dy\right| \le \left(\int_K |E(y)|^2\,dy\right)^{1/2} \|\Delta(\chi u)\|_{L^2}.$$

Thus, $|\chi u(x)|^2 \le C\|\Delta(\chi u)\|_{L^2}^2$, so dividing by $|x|^2$ and integrating on both sides, we get

$$\begin{aligned}\int \frac{|\chi u(x)|^2}{|x|^2} &\le C\|\Delta(\chi u)\|_{L^2} \\ &\le C(\|\Delta u\|^2 + \|u\|^2 + \|\nabla u\|^2) \\ &\le C'(\|\Delta u\|^2 + \|u\|^2).\end{aligned}$$

## 15.2 Quadratic forms

Let $H$ be a complex, separable Hilbert space, let $D \subseteq H$ be a linear subspace, and let $q : D \times D \to \mathbb{C}$ be a sesquilinear form. Let $q(u) := q(u,u)$ be the corresponding quadratic form with domain $D(q) = D$.

**Remark 15.1.** The polarization identity

$$q(u,v) = \frac{1}{4}\sum_{k=0}^{3} i^k q(u + i^k v)$$

allows us to determine $q(u,v)$ from $q(u)$.

**Definition 15.1.** We say that $q$ is **symmetric** if $q(u,v) = \overline{q(v,u)}$ for all $(u,v) \in D$ (so $q(u) \in R$ for all $u$). A symmetric form $q$ is **bounded below** if $q(u) \geq -C\|u\|^2$ for all $u \in D(q)$.

**Example 15.2.** Let $H = L^2(\mathbb{R}/2\pi\mathbb{Z})$, and let $V \in L^1(\mathbb{T};\mathbb{R})$. Consider $q(u) = \int_{\mathbb{R}/2\pi\mathbb{Z}}(|u'|^2 + V|u|^2)\,dx$ with domain $D(q) = H^1(\mathbb{T}) \subseteq L^\infty(\mathbb{T})$. Formally, we can write

$$q(u) = \langle Pu, u\rangle_{L^2}\,, \qquad P = -\partial_x^2 + V.$$

We'll apply quadratic form techniques to discuss self-adjoint extensions.

# 16 Closure of Quadratic Forms

## 16.1 Quadratic forms bounded below

Last time, we introduced the notion of quadratic forms: Let $q : D \to \mathbb{R}$ be a symmetric quadratic form. We say that $q$ is **bounded below** if there is a $c$ such that $q(u) \geq -c\|u\|^2$ for $u \in D$.

**Example 16.1.** Let $H = L^2(\mathbb{T})$, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, and $V \in L^1(\mathbb{T};\mathbb{R})$. Let

$$q(u) = \int (|u'|^2 + V|u|^2)\,dx, \qquad D = D(q) = H^1(\mathbb{T}) \subseteq L^\infty(\mathbb{T}).$$

Formally, $q(u) = \langle Pu, u\rangle_{L^2}$ with $P = -\partial_x^2 + V(x)$.

We claim that $q$ is bounded below: For every $\varepsilon > 0$, there is some $V^\sharp \in L^\infty(\mathbb{T})$ such that $\|V - V^\sharp\|_{L^1} \leq \varepsilon$. Then, keeping in mind that $\|u\|_{H_1}^2 = \|u'\|_{L^2}^2 + \|u\|_{L^2}^2$,

$$\begin{aligned} q(u) &= \int |u'|^2 + \underbrace{\int V^\sharp |u|^2}_{\geq -C_\varepsilon \|u\|_{L^2}^2} + \underbrace{\int (V - V^\sharp)|u|^2}_{\geq -O(\varepsilon)\|u\|_{H^1}^2} \\ &\geq (1 - O(\varepsilon))\|u\|_{H^1}^2 - C_\varepsilon \|u\|_{L^2}^2 \end{aligned}$$

for all $\varepsilon > 0$. We get that there exist $c > 0, C > 0$ such that

$$q(u) \geq c\|u\|_{H^1}^2 - C\|u\|_{L^2}^2, \qquad \forall u \in D(q).$$

**Remark 16.1.** We can always add a constant multiple of $\|u\|$ to $q$, so from now on, we will assume that $q$ is nonnegative. This allows us to use the Cauchy-Schwarz inequality:

$$|q(u,v)| \leq q(u)^{1/2} q(v)^{1/2} \qquad \forall u, v \in D(q).$$

## 16.2 Closed quadratic forms

**Definition 16.1.** Let $u_n \in D(q)$ and $u \in H$. We say that $u_n$ is **$q$-convergent** to $u$ (written $u_n \xrightarrow{q} u$) if $u_n \to u$ in $H$ and $q(u_n - u_m) \xrightarrow{n,m\to\infty} 0$. We say that $q$ is **closed** if whenever $u_n \xrightarrow{q} u$, $u \in D(q)$ and $q(u_n - u) \to 0$.

**Remark 16.2.** Let $H_q = D(q)$, equipped with the scalar product $\langle u, v\rangle_q := q(u,v) + \langle u, v\rangle$. Then $q$ is closed if and only if $H_q$ is a Hilbert space.

Let's return to our example.

**Example 16.2.** Let $q(u) = \int(\|u'|^2 + V|u|^2)$ on $D(q) = H^1(\mathbb{T})$ with $V \in L^1(\mathbb{T})$. Then $q(u) \geq c\|u\|_{H^1}^2 - C\|u\|_{H^1}^2$, so the quadratic form $u \mapsto q(u) + C\|u\|_{H^1}^2$ is closed. Indeed,

$$\|u\|_q^2 := q(u) + (c+1)\|u\|_{H^1}^2,$$

and this inequality tells us that $\|u\|_q \sim \|u\|_{H^1}$ for $u \in D(q)$.

### 16.3 Closable quadratic forms

**Definition 16.2.** We say that $q$ is **closable** if it has a closed extension $\widetilde{q} : D(\widetilde{q}) \to \mathbb{R}$: $D(q) \subseteq D(\widetilde{q})$ and $\widetilde{q}|_{D(q)} = q$.

**Proposition 16.1.** *A quadratic form $q$ is closable if and only if whenever $u_n \xrightarrow{q} 0$, then $q(u_n) \to 0$. If this condition holds, then $q$ has a smallest closed extension $\overline{q}$ (the* ***closure*** *of $q$) given by $D(\overline{q}) = \{u \in H : \exists u_n \in D(q) \text{ s.t. } u_n \xrightarrow{q} u\}$ and $\overline{q}(u) = \lim_{n\to\infty} q(u_n)$.*

*Proof.* ($\implies$): Let $\widetilde{q}$ be a closed extension. If $u_n \xrightarrow{q} 0$, then $u_n \xrightarrow{\widetilde{q}} 0$, so $\widetilde{q}(u_n) = q(u_n) \to 0$.

($\impliedby$): Assume that the condition holds, and let $u_n \xrightarrow{q} u$. We claim that $\lim_{n\to\infty} q(u_n)$ exists.

$$\begin{aligned}|q(u_n) - q(u_m)| &= |q(u_n, u_n) - q(u_m, u_m)| \\ &= |q(u_n - u_m, u_m) + q(u_m, q_n - u_m) \\ &\overset{\text{C-S}}{\leq} q^{1/2}(u_n - u_m) q^{1/2}(u_n) + q^{1/2}(u_n - u_m) q^{1/2}(u_m).\end{aligned}$$

So we get

$$|q^{1/2}(u_n) - q^{1/2}(u_m)| \leq q^{1/2}(u_n - u_m) \xrightarrow{n,m\to\infty} 0.$$

The claim follows, and if $v_n \xrightarrow{q} u$, then $u_n - v_n \xrightarrow{q} 0$:

$$q^{1/2}(u_n - v_n - u_m + v_m) \leq q^{1/2}(u_n - u_m) + q^{1/2}(v_n - v_m) \to 0.$$

By the assumed condition, $q(u_n - v_n) \to 0$. And by the same argument as before,

$$|q^{1/2}(u_n) - q^{1/2}(v_n)| \leq q^{1/2}(u_n - v_n) \to 0.$$

We get a well-defined quadratic form $\overline{q}$ which extends $q$.

We claim that $\overline{q}$ is closed; that is, we check that $H_{\overline{q}} = D(\overline{q})$ is complete with respect to $\langle u, v\rangle_{\overline{q}} = \overline{q}(u, v) + \langle u, v\rangle$. $H_q$ is dense in $H_{\overline{q}}$, as if $u_n \in H_q$ with $u_n \xrightarrow{q} u \in H_{\overline{q}}$, then $\overline{q}(u_n - u) \to 0$:

$$\overline{q}(u_n - u) = \lim_{m\to\infty} q(u_n - u_m) \xrightarrow{n\to\infty} 0.$$

Every Cauchy sequence in $H_q$ has a limit in $H_{\overline{q}}$, so we have a dense subset where every Cauchy sequence has a limit. So $H_{\overline{q}}$ is complete.

Finally, one checks that if $\widetilde{q}$ is a closed extension of $q$, then $\overline{q} \subseteq \widetilde{q}$. $\square$

**Theorem 16.1.** *Let $q$ be a nonnegative, symmetric, quadratic form. Assume that $D(q)$ is dense and that $q$ is closed. Then there exists a unique self-adjoint operator $\mathscr{A}$ such that $D(\mathscr{A}) \subseteq D(q)$ and $q(u, v) = \langle \mathscr{A} u, v\rangle$ for all $u \in D(\mathscr{A}), v \in D(q)$. Also, $D(\mathscr{A})$ is a* ***core*** *for $q$ in the sense that $D(\mathscr{A})$ is dense in $H_q$.*

# 17 Quadratic Forms and the Friedrichs Extension Theorem

## 17.1 Obtaining self-adjoint operators from quadratic forms

Last time, we said that a nonnegative, symmetric quadratic form is closed if when $u_n \xrightarrow{q} u$ for $u_n \in D(q)$, then $u \in D(q)$ and $q(u_n - u) \to 0$. We checked that $q$ is closable if and only if when $u_n \xrightarrow{q} 0$, $q(u_n) \to 0$.

**Theorem 17.1.** *Let $q$ be a nonnegative, symmetric, quadratic form. Assume that $D(q)$ is dense and that $q$ is closed. Then there exists a unique self-adjoint operator $\mathscr{A}$ such that $D(\mathscr{A}) \subseteq D(q)$ and $q(u,v) = \langle \mathscr{A}u, v\rangle$ for all $u \in D(\mathscr{A}), v \in D(q)$. Also, $D(\mathscr{A})$ is a* ***core*** *for $q$ in the sense that $D(\mathscr{A})$ is dense in $H_q$.*

**Example 17.1.** Let $q(u) = \int |u'|^2 + V|u|^2\,dx$, where $V \in L^1(\mathbb{T};\mathbb{R})$ and $D(q) = H^1(\mathbb{T})$. Then there exists a unique self-adjoint operator $P = -\partial_x^2 + V$ such that $D(P) \subseteq H^1$ and $q(u,v) = \langle Pu, v\rangle$.

*Proof.* Let $\langle x, y\rangle_q := q(x,y) + \langle x, y\rangle$ for $x, y \in D(q)$. Then $H_q$ is a Hilbert space with respect to this scalar product. Then $\|x\|_q \geq \|x\|$ for all $x \in H_q$, so for any $u \in H$, the linear form $H_q \to \mathbb{C}$ sending $v \mapsto \langle v, u\rangle$ is continuous. By the Riesz representation theorem, there is a unique $u^* \in H_q$ such that $\langle v, u\rangle = \langle v, u^*\rangle_{H_q}$. We get a linear map $K : H \to H_q$ sending $u \mapsto u^*$ such that $\langle v, u\rangle = \langle v, Ju\rangle_q$ for all $v \in H_q$ and $u \in H$.

We claim that $J$ is a bounded, self-adjoint operator on $H$. If $y, x \in H$,

$$\langle Jy, x\rangle = \langle Jy, Jx\rangle_q = \overline{\langle Jx, Jy\rangle_q} = \overline{\langle Jx, y\rangle} = \langle y, Jx\rangle.$$

So $J$ is symmetric. By the closed graph theorem, $J \in \mathcal{L}(H,H)$. Moreover, $J$ is injective: If $Jx = 0$, then $\langle y, x\rangle = \langle y, Jx\rangle_q = 0$ for all $y \in H_q$. But $H_q$ is dense in $H$, so $x = 0$.

Write

$$H = \ker J \oplus \overline{\operatorname{Ran} J^*} = \overline{\operatorname{Ran} J}.$$

So $\operatorname{Ran} J$ is dense and contained in $H_q$. Define $\mathscr{A} : D(\mathscr{A}) = \operatorname{Ran} J$ by $\mathscr{A}x = J^{-1}x - x$. We have $J^{-1}$ is self-adjoint: $J^{-1}$ is symmetric, and if $(x,y)$ are such that $\langle J^{-1}z, x\rangle = \langle z, y\rangle$, where $z = Jw$, then $\langle w, x\rangle = \langle Jw, y\rangle = \langle w, Jy\rangle$. So $x \in D(\mathscr{A})$, and $y = J^{-1}x$. ($D(\mathscr{A})$ is dense in $H_q$.)

Finally, $\mathscr{A}$ corresponds to the quadratic form $q$:

$$\langle x, y\rangle_q = \langle x, J^{-1}y\rangle,$$

so

$$q(x,y) = \langle x, y\rangle_q - \langle x, y\rangle = \langle x, \mathscr{A}y\rangle, \qquad y \in D(\mathscr{A}), x \in D(q).$$ □

## 17.2 The Friedrichs extension theorem

In the previous theorem, we don't have much control over the domain of the self-adjoint operator $\mathscr{A}$. Here is a frequently encountered use of the theorem.

**Theorem 17.2** (Friedrichs extension)**.** *Let $S$ be a symmetric, densely defined operator $D(S) \to H$ such that $S$ is bounded below: $\langle Su, u\rangle \geq -C\|u\|^2$ for every $u \in D(S)$. Let the quadratic form $q$ be given by $D(q) = D(S)$, $q(u) = \langle Su, u\rangle$. Then $q$ is closable. The self-adjoint operator associated to $\overline{q}$, the **Friedrichs extension** of $S$, is also bounded below.*

**Remark 17.1.** The Friedrichs extension theorem can give a different result compared to if we just closed the operator $S$.

**Remark 17.2.** Let $q \geq 0$ be closed. Then $q(u,v) = \langle u, \mathscr{A}v\rangle$ for $v \in D(\mathscr{A})$ and $u \in D(q)$, where $D(\mathscr{A}) = \{v \in H_q : \exists f \in H \text{ s.t. } q(u,v) = \langle u, f\rangle \ \forall u \in D(q)\}$. So in the theorem, $D(\mathscr{A}) \supseteq D(S)$.

*Proof.* We can assume that $S \geq 0$. We only need to show that $q$ is closable. Let $u_n \in D(S)$ be such that $u_n \to 0$ in $H$ and $q(u_n - u_m) \xrightarrow{n,m\to\infty} 0$. We want to show that $q(u_n) \to 0$. We have

$$\begin{aligned} q(u_n) &\leq |q(u_n - u_m, u_n)| + |q(u_m, u_n)| \\ &\overset{\text{C-S}}{\leq} q^{1/2}(u_n - u_m) q^{1/2}(u_n) + |\langle u_m, Su_n\rangle|. \end{aligned}$$

For all $\varepsilon > 0$, there exists an $N$ such that $q(u_n - u_m) \leq \varepsilon$ when $n, m \geq N$. So

$$q(u_n) \leq \varepsilon^{1/2} q^{1/2}(u_n) + |\langle u_m, Su_n\rangle| \qquad \forall n, m \geq N.$$

Leting $m \to \infty$, we get

$$q(u_n) \leq \varepsilon \qquad \forall n \geq N.$$

So $q$ is closable. $\square$

**Example 17.2** (The Dirichlet realization of $-\Delta$)**.** Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded, and let $S = -\Delta$ with $D(S) = C_0^\infty(\Omega)$ ($S \geq 0$). The Friedrichs extension is associated with the closure of the quadratic form

$$q(u) = \langle Su, u\rangle = \int_\Omega (-\Delta u)\overline{u} = \int_\Omega |\nabla u|^2 \, dx.$$

Next time, we will see that in this case, $D(\overline{q})$ is the closure of $C_0^\infty$ in the topology of $H^1(\Omega)$. This is usually called $H_0^1(\Omega)$ (but is not all of $H^1(\Omega)$).

# 18 Introduction to Spectral Theory of Unbounded Operators

## 18.1 The Dirichlet realization of a 2nd order elliptic operator

Let $q : D(q) \to \mathbb{R}$ be a nonnegative, symmetric, densely defined, closed quadratic form. last time, we saw that there is a unique self-adjoint operator $\mathscr{A}$ with $D(\mathscr{A}) \subseteq D(q)$, $q(u,v) = \langle u, \mathscr{A} v\rangle$ for $u \in D(q)$ and $v \in D(\mathscr{A})$. We have

$$D(\mathscr{A}) = \{v \in D(q) : \exists f \in H \text{ s.t. } q(u,v) = \langle u, f\rangle \ \forall u \in D(q)\}.$$

**Example 18.1** (Dirichlet realization of a 2nd order elliptic operator)**.** Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded, and let $\Omega \ni x \mapsto (a_{j,k}(x)) \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ with $a_{j,k} = a_{k,j}$ with $a_{j,k} \in L^\infty(\Omega)$. Assume the **ellipticity condition**:

$$\exists c > 0 \text{ such that} \qquad \sum_{j,k=1}^n a_{j,k}(x)\xi_j\xi_k \geq c|\xi|^2 \qquad \forall x \in \Omega, \xi \in \mathbb{R}^n.$$

Let

$$q(u) = \int_\Omega \sum_{j,k=1}^n a_{j,k}(x) \frac{\partial u}{\partial x_j} \overline{\frac{\partial u}{\partial x_k}}\, dx,$$

where $D(q) = H_0^1(\Omega)$, the closure of $C_0^\infty(\Omega)$ in the Sobolev space $H^1(\Omega) = \{u \in L^2(\Omega) : \partial_{x_j} u \in L^2\}$. Then $q$ is closed: $q(u) + \|u\|_{L^2}^2$ is equivalent to $\|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2$ on $H_0^1(\Omega)$.

Associated to $q$ is a self-adjoint operator $\mathscr{A}$ with

$$\begin{aligned} D(\mathscr{A}) &= \{u \in H_0^1 : \exists f \in L^2 \text{s.t. } q(u,v) = \langle f, v\rangle \ \forall v \in H_0^1(\Omega)\} \\ &= \{u \in H_0^1 : \exists f \in L^2 \text{s.t. } q(u,v) = \langle f, v\rangle \ \forall v \in H_0^1(\Omega)\} \end{aligned}$$

Rewrite this condition: $\int \sum a_{j,k} \frac{\partial u}{\partial x_j} \overline{\frac{\partial v}{\partial x_k}} = \int f\overline{v} \,\forall v \iff \sum_{j,k=1}^n \partial_{x_j}(a_{j,k}(x)\frac{\partial u}{\partial x_k}) \in L^2(\Omega)$.

$$= \left\{u \in H_0^1 : \sum \partial_{x_j}\left(a_{j,k}\frac{\partial u}{\partial x_k}\right) \in L^2\right\}.$$

$\mathscr{A}$ is given by

$$\mathscr{A} u = -\sum \partial_{x_j}\left(a_{j,k}\frac{\partial u}{\partial x_k}\right), \qquad u \in D(\mathscr{A}).$$

The operator $\mathscr{A}$ is the **Dirichlet realization** of $-\sum_{j,k=1}^n \partial_{x_j}(a_{j,k}\frac{\partial u}{\partial x_k})$.

We have

$$\langle \mathscr{A} u, u\rangle = q(u) = \int \sum a_{j,k} \frac{\partial u}{\partial x_j}\overline{\frac{\partial u}{\partial x_k}} \geq c\|\nabla u\|_{L^2}^2 \geq c'\|u\|_{L^2}^2,$$

where the last inequality is **Poincaré's inequality**. So we get $\mathscr{A} : D(\mathscr{A}) \to L^2(\Omega)$ is bijective: $\mathscr{A}$ is injective, $\operatorname{im} A$ is closed (by Cauchy-Schwarz), and $(\operatorname{im} \mathscr{A})^\perp = \ker \mathscr{A} = \{0\}$.

**Remark 18.1.** When $(a_{j,k}) = 1$, the corresponding operator is $\mathscr{A} = -\Delta_D$ with $(-\Delta_D) = \{u \in H_0^1 : \Delta u \in L^2\}$. One can show that if $\partial\Omega \in C^2$, then $D(-\Delta_D) = (H_0^1 \cap H^2)(\Omega)$.

Let's look at spectral properties of $\mathscr{A}$.

$\mathscr{A}^{-1}$ is bounded: $L^2(\Omega) \to L^2(\Omega)$, and it is also bounded $L^2(\Omega) \to D(\mathscr{A})$ (equipped with the graph norm). There is a natural embedding $D(\mathscr{A}) \to H_0^1(\Omega) \to L^2(\Omega)$, where the embedding $H_0^1(\Omega) \to L^2(\Omega)$ is compact (Rellich compactness theorem). Thus, $\mathscr{A}^{-1}$ is compact, and self-adjoint on $L^2(\Omega)$. If $\lambda_1 \geq \lambda_2 \geq \cdots \to 0$ are the nonvanishing eigenvalues of $\mathscr{A}^{-1}$ (each eigenvalue repeated according to its multiplicity), then let $e_n \in L^2(\Omega)$ be the corresponding eigenfunctions: $(\mathscr{A}^{-1} - \lambda_n)e_n = 0$. The $e_n$ form an orthonormal basis of $L^2(\Omega)$; moreover, $e_n \in D(\mathscr{A})$, and $(\mathscr{A} - 1/\lambda_n)e_n = 0$.

Now for $z \in \mathbb{C}$, $\mathscr{A} - z : D(\mathscr{A}) \to L^2(\Omega)$ is invertible if and only if $z \neq 1/\lambda_n$ for all $n$:

$$\mathscr{A} - z = \underbrace{(1 - z\mathscr{A}^{-1})}_{\text{invertible iff injective}} \mathscr{A},$$

and the first term is injective iff $z \neq 1/\lambda_n$ for all $n$. We can conclude that the **spectrum** of $\mathscr{A}$ is given by the eigenvalues $\mu_1 \leq \mu_2 \leq \cdots \to \infty$ with $(\mathscr{A} - \mu_n)e_n = 0$ and $e_n$s forming an orthonormal basis for $L^2(\Omega)$.

## 18.2 Spectrum and resolvent

**Definition 18.1.** Let $T : D(T) \to H$ be closed and densely defined. We say that for $\lambda \in \mathbb{C}$, $\lambda \notin \operatorname{Spec}(T)$ if and only if $T - \lambda : D(T) \to H$ is bijective. The complement of $\operatorname{Spec}(T)$ is the **resolvent set** of $T$. When $\lambda \notin \operatorname{Spec}(T)$, we let $R(\lambda) = (T - \lambda)^{-1}$ be the **resolvent** of $T$.

Since $T$ is closed, $R(\lambda)$ is closed. By the closed graph theorem, $R(\lambda) \in \mathcal{L}(H, H)$.

**Proposition 18.1.** *The resolvent set $\rho(T) \subseteq \mathbb{C}$ is open, and $\rho(T) \ni \lambda \mapsto R(\lambda) \in \mathcal{L}(H, H)$ is holomorphic.*

Next time we will prove this. We are working towards a development of the spectral theorem for unbounded operators.

# 19 Development Toward the Spectral Theorem for Unbounded Self-Adjoint Operators

## 19.1 The resolvent

Let $T : D(T) \to H$ be closed and densely defined. We said $\lambda \notin \operatorname{Spec}(T) \iff T - \lambda : D(T) \to H$ is bijective and defined the **resolvent** as $R(\lambda) = (T - \lambda)^{-1} \in \mathcal{L}(H, H)$ for $\lambda \in \rho(T) = \mathbb{C} \setminus \operatorname{Spec}(T)$, the **resolvent set**.

**Proposition 19.1.** *The resolvent set $\rho(T) \subseteq \mathbb{C}$ is open, and $\rho(T) \ni \lambda \mapsto R(\lambda) \in \mathcal{L}(H, H)$ is holomorphic.*

*Proof.* If $\lambda_0 \in \rho(T)$ write

$$T - \lambda = (T - \lambda) - (\lambda - \lambda_0) = (1 - (\lambda - \lambda_0) R(\lambda_0))(T - \lambda_0).$$

It follows that $T - \lambda$ is invertible for $|\lambda - \lambda_0| < \frac{1}{\|R(\lambda_0)\|}$. We also have

$$\begin{aligned} R(\lambda) = R(\lambda_0) &= R(\lambda_0)(1 - (\lambda - \lambda_0) R(\lambda_0))^{-1} \\ &= \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n R(\lambda_0)^{n+1}. \end{aligned}$$

This converges in $\mathcal{L}(H, H)$, so $R(\lambda)$ is holomorphic. $\square$

**Proposition 19.2.** *If $T$ is self-adjoint, then* $\operatorname{Spec}(T) \subseteq \mathbb{R}$.

*Proof.* If $\lambda \in \mathbb{C} \setminus \mathbb{R}$,

$$\|(T - \lambda)u\|^2 = \|(T - \operatorname{Re} \lambda)u\|^2 + (\operatorname{Im} \lambda)^2 \|u\|^2. \qquad \square$$

We also have

$$\|R(\lambda)\|_{\mathcal{L}(H,H)} \leq \frac{1}{|\operatorname{Im} \lambda|}.$$

## 19.2 Nevanlinna-Herglotz functions for self-adjoint operators

**Example 19.1.** Let $:D(\mathscr{A}) \to H$ be self-adjoint, $\langle \mathscr{A} u, u \rangle \geq c \|u\|^2$ for $u \in D(\mathscr{A})$, and $\mathscr{A}^{-1} : H \to H$ be compact. Then $\operatorname{Spec}(A)$ is of the form $\lambda_1 \leq \lambda_2 \leq \cdots \to \infty$. Let $e_j$ be an orthonormal basis of $H$ such that $(\mathscr{A} - \lambda_j) e_j = 0$. Then for $u = \sum_j x_j e_j \in H$,

$$\langle R(z) u, u \rangle = \sum \frac{|c_j|}{\lambda_j - z} = \int \frac{\xi - z}{} d\mu(\xi), \qquad d\mu = \sum |c_j|^2 \delta(\xi - \lambda)$$

This is a positive, pure point measure of total mass $\int d\mu = \|u\|^2$ (by Parseval).

Let $A : D(A) \to H$ be self-adjoint, and consider the holomorphic function

$$f(z) = \langle R(z)u, u\rangle, \qquad \operatorname{Im} z > 0.$$

Then $|f(z)| \le \frac{C}{|\operatorname{Im} z|}$, where $C = \|u\|^2$. We have

$$2i \operatorname{Im} f(z) = \langle R(z)u, u\rangle - \langle R(z)^* u, u\rangle$$

Using $R(z)^* = R(\overline{z})$,

$$= \langle (R(z) - R(\overline{z}))u, u\rangle$$

We can also check via algebraic manipulation that $R(\lambda) - \mathbb{R}(\mu) = (\lambda - \mu)R(\lambda)R(\mu)$. Using this,

$$= 2i \operatorname{Im} z \,\langle R(z)^* R(z)u, u\rangle.$$

So we get

$$\operatorname{Im} f = \operatorname{Im} z \|R(z)u\|^2.$$

In particular, $\operatorname{Im} f \ge 0$ for $\operatorname{Im} z > 0$.

Now we can use the following general result from complex analysis.[6]

**Theorem 19.1** (Nevanlinna, Herglotz,...). *Let $f$ be a holomorphic function in $\operatorname{Im} z > 0$ with $\operatorname{Im} f \ge 0$ and $|f(z)| \le \frac{c}{\operatorname{Im} z}$. Then there is a uniform bound*

$$\int \operatorname{Im} f(x+iy)\, dx \le C\pi \qquad \forall y > 0,$$

*and there exists a positive bounded measure $\mu$ on $\mathbb{R}$ such that*

$$\frac{1}{\pi} \int \varphi(x) \operatorname{Im} f(x+iy)\, dx \xrightarrow{y \to 0^+} \int \varphi \, d\mu \qquad \forall \varphi \in C_B := (C \cap L^\infty)(\mathbb{R}).$$

*We have*

$$f(z) = \int \frac{1}{\xi - z} d\mu(\xi), \qquad \operatorname{Im} z > 0,$$

$$\int d\mu(\xi) = \lim_{y \to +\infty} y \operatorname{Im} f(iy) = \lim_{z \to \infty} (-z f(z)),$$

*where $z \to \infty$ with $\arg(z)$ bounded away from $0, \pi$.*

*Conversely, if $\mu \ge 0$ is a bounded measure on $\mathbb{R}$ and $f$ is defined by $f(z) = \int \frac{1}{\xi - z} d\mu(\xi)$, then both the weak convergence and the limit condition hold.*

[6] This is a standard result. It was even a qualifying exam problem in the past.

*Proof.* Take the semicircle contour $\gamma$ (slightly above the real axis) made of $\operatorname{Im} z = c > 0$ and an arc of radius $R$. Cauchy's integral formula gives

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \\ &= \frac{1}{2\pi i} \int_\gamma f(\zeta) \left( \frac{1}{\zeta - z} - \frac{1}{\zeta - z^*} \right) d\zeta, \end{aligned}$$

where $z^*$ is the regkection of $z$ over the line $\operatorname{Im} z = c$. That is, $z^* = \overline{z} + 2ic$.

$$= \frac{1}{2\pi i} \int_\gamma f(\zeta) \frac{z - z^*}{(\zeta - z)(\zeta - z^*)} \, d\zeta$$

Letting $R \to \infty$, we get

$$f(z) = \frac{1}{2\pi i} \int_{L_c} \frac{f(\zeta)(z - z^*)}{(\zeta - z)(\zeta^* - z^*)} \, d\zeta,$$

where $L_c$ is the whole line $\operatorname{Im} z = c$ and $(\zeta - z)(\zeta^* - z^*) = |\zeta - z|^2$. Take the imaginary part of this to get

$$\operatorname{Im} f = \frac{\operatorname{Im} z - c}{\pi} \int_{L_c} \frac{\operatorname{Im} f(\zeta)}{|\zeta - z|^2} \, d\zeta$$

Multiply by $\operatorname{Im} z$ and let $\operatorname{Im} z \to \infty$ while keeping $\operatorname{Re} z$ fixed: the left hand side is $\leq c$. By Fatou's lemma,

$$\frac{1}{\pi} \int_{\mathbb{R}} \operatorname{Im} f(x + ic) \, dx \leq c.$$

By Banach-Alaoglu, there is a sequence of $c_n \to 0^+$ and a positive, bounded measure $\mu$ on $\mathbb{R}$ such that

$$\frac{1}{\pi} \operatorname{Im} f(x + ic_n) \, dx \xrightarrow{\text{weak*}} \mu.$$

We get that

$$\operatorname{Im} f(z) = \operatorname{Im} z \int \frac{1}{|\xi - z|^2} \, d\mu(\xi), \qquad \operatorname{Im} z > 0.$$

This implies that

$$\operatorname{Im} \left( f(z) - \int \frac{1}{\xi - z} \, d\mu(\xi) \right) = 0,$$

so

$$f(z) = \int \frac{1}{\xi - z} \, d\mu(\xi),$$

proving the first claim. $\square$

We will finish the proof of the theorem next time.

# 20 Development of Spectral Measures

## 20.1 Nevanlinna-Herglotz functions

Last time, we were proving the following theorem from complex analysis.

**Theorem 20.1** (Nevanlinna, Herglotz,...). *Let $f$ be a holomorphic function in $\operatorname{Im} z > 0$ with $\operatorname{Im} f \geq 0$ and $|f(z)| \leq \frac{c}{\operatorname{Im} z}$. Then there is a uniform bound*

$$\int \operatorname{Im} f(x+iy)\, dx \leq C\pi \qquad \forall y > 0,$$

*and there exists a positive bounded measure $\mu$ on $\mathbb{R}$ such that*

$$\frac{1}{\pi} \int \varphi(x) \operatorname{Im} f(x+iy)\, dx \xrightarrow{y \to 0^+} \int \varphi\, d\mu \qquad \forall \varphi \in C_B := (C \cap L^\infty)(\mathbb{R}).$$

*We have*

$$f(z) = \int \frac{1}{\xi - z} d\mu(\xi), \qquad \operatorname{Im} z > 0,$$

$$\int d\mu(\xi) = \lim_{y \to +\infty} y \operatorname{Im} f(iy) = \lim_{z \to \infty} (-zf(z)),$$

*where $z \to \infty$ with $\arg(z)$ bounded away from $0, \pi$.*

*Conversely, if $\mu \geq 0$ is a bounded measure on $\mathbb{R}$ and $f$ is defined by $f(z) = \int \frac{1}{\xi - z} d\mu(\xi)$, then both the weak convergence and the limit condition hold.*

Last time, we showed that $f(z) = \int \frac{1}{\xi - z} d\mu(\xi)$, which implies that

$$\int d\mu(\xi) = \lim_{y \to \infty} y \operatorname{Im} f(iy) = \lim_{z \to \infty} (-zf(z)).$$

*Proof.* We claim that $\frac{1}{\pi} \operatorname{Im} f(x+iy)\, dx \to d\mu$ weak* as $y \to 0^+$. If $\varphi \in C_B(\mathbb{R})$,

$$\begin{aligned}
\frac{1}{\pi} \operatorname{Im} f(x+iy) \varphi(x)\, dx &= \frac{1}{\pi} \iint \varphi(x) \frac{y}{|\xi - x - iy|^2} d\mu(\xi)\, dx \\
&= \int \left( \frac{y}{\pi} \int \frac{\varphi(x)}{|\xi - x - iy|^2}\, dx \right) d\mu(\xi)
\end{aligned}$$

The part inside the parentheses is $\frac{1}{\pi} \int \frac{\varphi(\xi + ty)}{1+t^2}\, dt \to \varphi(\xi)$ by cominated convergence as $y \to 0^+$. The convergence is dominated by $\|\varphi\|_{L^\infty} \in L^1(d\mu)$, so by another application of dominated convergence, we get

$$\xrightarrow{y \to 0^+} \int \varphi(\xi)\, d\mu(\xi). \qquad \square$$

## 20.2 Construction of spectral measures via Nevanlinna-Herglotz functions

Apply the theorem to $f(z) = \langle R(z)u, u\rangle$, where $u \in H$ and $R(z) = (A-z)^{-1}$ is the resolvent of a self-adjoint operator $A$. We can write

$$\langle R(z)u, u\rangle = \int \frac{1}{\xi - z}\, d\mu_u(\xi),$$

where the total measure is

$$\int d\mu_u = \lim_{y\to+\infty} y \underbrace{\operatorname{Im}\langle R(iy)u, u)\rangle}_{y\|R(iy)u\|^2} \le \|u\|^2.$$

We have, for $\varphi \in C_B$,

$$\begin{aligned}\int \varphi(x)\, d\mu_u(x) &= \lim_{y\to 0^+} \frac{1}{\pi} \int \operatorname{Im}\langle R(x+iy)u, u\rangle\, \varphi(x)\, dx \\ &= \lim_{y\to 0^+} \frac{1}{2\pi i} \varphi(x)[\langle R(x+iy)u, u\rangle - \langle R(x-iy)u, u\rangle]\, dx\end{aligned}$$

So the measure is given by the jump of the resolvent over the real line.

**Remark 20.1.** We have that $\operatorname{supp}(\mu_n) \subseteq \operatorname{Spec}(A)$ for all $u$.

Define the complex measures $\mu_{u,v}$ for $u, v \in H$ by polarization:

$$d\mu_{u,v} = \frac{1}{4}(d\mu_{u+v} + id\mu_{u+iv} - d\mu_{u-v} - id\mu_{u-iv}).$$

Then we have

$$\langle R(z)u, v\rangle = \int \frac{1}{\xi - z}\, d\mu_{u,v}(\xi),$$

$$\int \varphi\, d\mu_{u,v} = \lim_{y\to 0^+} \frac{1}{2\pi i} \int \varphi(x)\, \langle (R(x+iy) - R(x-iy))u, v\rangle\ dx.$$

The total mass of $\mu_{u,v}$ is

$$\int d\mu_{u,v} \le \frac{1}{4}(\|u+v\|^2 + \|u+iv\|^2 + \|u-v\|^2 + \|u-iv\|$$

By homogeneity, we may assume $\|u\| = \|v\| = 1$.

$$= \frac{1}{4}(2(\|u\|^2 + \|v\|^2)2) = 2.$$

So by homogeneity, $\|\mu_{u,v}\| \le 2\|u\| \cdot \|v\|$.

For fixed $\varphi \in C_B$, we have

$$\left| \int \varphi \, d\mu_{u,v} \right| \leq 2\|\varphi\|_{L^\infty} \|u\| \|v\|,$$

so the map $(u, v) \mapsto \int \varphi \, d\mu_{u,v}$ is a bounded, sesquilinear form on $H \times H$. By the Riesz representation theorem, there is a unique operator $\varphi(A) \in \mathcal{L}(H, H)$ such that $\langle \varphi(A)u, v \rangle = \int \varphi \, d\mu_{u,v}$. We also get that $\|\varphi(A)\| \leq 2\|\varphi\|_{L^\infty}$. In other words,

$$\varphi(A) = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int \varphi(\lambda)(R(\lambda + i\varepsilon) - R(\lambda - i\varepsilon)) \, d\lambda$$

(where the limit exists in the weak operator topology).

**Remark 20.2.** Let $A : \mathbb{C}^n \to \mathbb{C}^n$ be a Hermitian matrix. Then the above relation holds: it suffices to check this formula when $n = 1$, where $A$ is multiplication by $t$ for $t \in \mathbb{R}$. In this case, this is

$$\varphi(t) = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int \varphi(\lambda) \left( \frac{1}{t - \lambda - i\varepsilon} - \frac{1}{t - \lambda + i\varepsilon} \right) d\lambda.$$

This is equivalent to

$$\delta(t) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \left( \frac{1}{t - i\varepsilon} - \frac{1}{t + i\varepsilon} \right),$$

which is called **Plemelj's formula**. The proof is (if we take $t = 0$):

$$\varphi(0) = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int \varphi(t) \frac{2i\varepsilon}{t^2 + \varepsilon^2} = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int \frac{\varepsilon \varphi(t)}{\varepsilon^2 + t^2}.$$

One sometimes writes

$$\delta(A - \lambda) = \frac{1}{2\pi i}(R(\lambda + i0) - R(\lambda - i0)),$$

$$\varphi(A) = \int \varphi(\lambda) \delta(A - \lambda) \, d\lambda.$$

**Definition 20.1.** The measures $\mu_{u,v}$ are called the **spectral measures** of $A$.

We have defined a map of algebras $C_B \to \mathcal{L}(H, H)$ sending $\varphi \mapsto \varphi(A)$. This map has nice algebraic properties that we will study. This will lead us to the development of the spectral theorem.

# 21 Properties of Spectral Measures

## 21.1 Total mass of spectral measures

Let $\varphi \in (C \cap L^\infty)(\mathbb{R})$ and let $A : D(A) \to H$ be self-adjoint. Last time, we had

$$\langle \varphi(A)u, v\rangle = \lim_{\varepsilon\to 0^+} \frac{1}{2\pi i}\int \varphi(\lambda)\,\langle (R(\lambda+i\varepsilon) - R(\lambda - i\varepsilon))u, v\rangle \, d\lambda.$$

This is similar to the finite dimensional case, where

$$\varphi(A) = \sum_{\lambda\in \operatorname{Spec}(A)} \varphi(\lambda)\Pi_\lambda,$$

where $\Pi_\lambda$ is the orthogonal projection on to $\ker(A-\lambda)$.

**Remark 21.1.** Observe that $\varphi(A)^* = \overline{\varphi}(A)$:

$$\begin{aligned}\langle \varphi(A)^* u, v\rangle &= \langle u, \varphi(A)v\rangle \\ &= \overline{\langle \varphi(A)v, u\rangle} \\ &= -\frac{1}{2\pi i}\lim_{\varepsilon\to 0^+}\int \overline{\varphi(\lambda)\langle (R(\lambda+i\varepsilon) - R(\lambda-i\varepsilon))u, v\rangle}\, d\lambda\end{aligned}$$

Using $R(\lambda \pm i\varepsilon) - R(\lambda \mp i\varepsilon)$,

$$= \langle \overline{\varphi}(A)u, v\rangle\,.$$

We also introduced the spectral measures $d\mu_{u,v}$ with $\langle \varphi(A)u, v\rangle = \int \varphi(\lambda)\, d\mu_{u,v}(\lambda)$.

**Proposition 21.1.** *The total mass of the spectral measure $d\mu_{u,v}$ is*

$$\int d\mu_{u,v} = \langle u, v\rangle\,.$$

*Equivalently,* $1(A) = I \in \mathcal{L}(H,H)$.

*Proof.* By continuity and density, we may assume $u, v$ are in a dense subset of $H$; assume $u, v \in D(A)$. By polarization, we may take $u = v$. If $u \in D(A)$, then

$$(A - z)u = Au - zu$$

If $\operatorname{Im} z > 0$, then we get

$$u = R(z)Au - zR(z)u,$$

so we get

$$R(z)u = -\frac{1}{z}u + \frac{1}{z}R(z)Au.$$

If $z \to \infty$ with $\operatorname{Re} z$ fixed, we get $R(z)u = -\frac{1}{z}u + O(1/|z|^2)$. Recall from Nevannlinna's theorem that

$$\int d\mu_u = \lim_{z\to\infty}(-z\,\langle R(z)u, u\rangle) = \|u\|^2.$$ □

## 21.2 Decay of spectral measures

**Proposition 21.2.** *Let $\varphi \in C_B$ For all $u \in D(A)$ and $v \in H$,*

$$\langle \varphi(A)Au, v \rangle = \langle \varphi_1(A)u, v \rangle ,$$

*where $\varphi_1(\lambda) = \lambda\varphi(\lambda) \in C_B$.*

*Proof.* The left hand side is

$$\text{LHS} = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int \varphi(\lambda) \left\langle (R(\lambda + i\varepsilon) - R(\lambda - i\varepsilon))u, v \right\rangle$$

Note that $(R(\lambda+i\varepsilon) - R(\lambda-i\varepsilon))u = u + (\lambda+i\varepsilon)R(\lambda+i\varepsilon)u - (\lambda-i\varepsilon)R(\lambda-i\varepsilon)u$.

$$= \langle \varphi_1(A)u, v \rangle + \lim_{\varepsilon \to 0^+} \frac{i\varepsilon}{2\pi i} \int \varphi(\lambda) \left\langle (R(\lambda + i\varepsilon) + R(\lambda - i\varepsilon))u, v \right\rangle \, d\lambda$$

To show that the right term equals 0, we have

$$O(\varepsilon) \left| \int \varphi(\lambda) \left\langle R(\lambda \pm i\varepsilon u, v \right\rangle \, d\lambda \right| \leq O(\varepsilon) \int |\varphi(\lambda)| \|R(\lambda \pm i\varepsilon)u\| \, d\lambda$$

By Cauchy-Schwarz,

$$\leq O(\varepsilon) \left( \int \|R(\lambda \pm i\varepsilon u)\|^2 \, d\lambda \right)$$

Recall that $\operatorname{Im} \langle R(\lambda + i\varepsilon)u, u \rangle = \varepsilon \|R(\lambda + i\varepsilon)u\|^2$.

$$\begin{aligned} &\leq O(\varepsilon^{1/2}) \left( \int \operatorname{Im} \langle R(\lambda + i\varepsilon)u, u \rangle \, d\lambda \right)^{1/2} \\ &\xrightarrow{\varepsilon \to 0} 0. \end{aligned}$$

We get that $\langle \varphi(A)Au, v \rangle = \langle \varphi_1(A)u, v \rangle$. $\square$

In particular, if $\varphi \in C_0(\mathbb{R})$, we have

$$\langle \varphi(A)Au, Au \rangle = \langle \varphi_1(A)u, Au \rangle = \langle u, \overline{\varphi}_1(A)Au \rangle = \left\langle u, \overline{\psi}(A)u \right\rangle ,$$

where $\psi(\lambda) = \lambda^2\varphi(\lambda)$. We get

$$\langle \varphi(A)Au, Au \rangle = \langle \psi(A)u, u \rangle .$$

On the level of spectral measures, we get

$$\int \varphi(\lambda) \, d\mu_{Au}(\lambda) = \int \lambda^2 \varphi(\lambda) \, d\mu_u(\lambda).$$

If $0 \le \varphi \le 1$, then the left hand side is $\le \|Au\|^2$. Letting $\varphi \uparrow 1$, we get by Fatou's lemma:

$$\int \lambda^2 \, d\mu_u(\lambda) < \infty.$$

By monotone convergence, we get

$$\int \lambda^2 \, d\mu_u(\lambda) = \|Au\|^2 < \infty \qquad \forall u \in D(A).$$

### 21.3 Multiplicativity of the functional calculus

**Proposition 21.3.** *Let $\varphi, \psi \in C_0(\mathbb{R})$. Then $\varphi(A)\psi(A) = (\varphi\psi)(A)$.*

*Proof.* Let $\varphi_k(\lambda) = \lambda^k \varphi(\lambda)$ for $k = 1, 2, \dots$. For $u \in H$ and $v \in D(A)$, we have:

$$\begin{aligned} \langle \varphi(A)u, Av \rangle &= \langle u, \overline{\varphi}(A)Av \rangle \\ &= \langle u, \overline{\varphi}_1(A)v \rangle \\ &= \langle \varphi_1(A)u, v \rangle . \end{aligned}$$

Thus, $\varphi(A)u \in D(A^*) = D(A)$ and $\varphi_1(A)u = A\varphi(A)u$. In particular, $\operatorname{im} \varphi(A) \subseteq D(A)$ for all $\varphi \in C_0$. So $\operatorname{im} \varphi_1(A) \subseteq D(A)$, so $\operatorname{im} \varphi(A) \subseteq D(A^2)$. Iterating this argument, we get that $\operatorname{im} \varphi(A) \subseteq D(A^j)$ for $j = 1, 2, \dots$ and $\varphi_j(A) = A^j \varphi(A)$ for any $j$. When $p$ is a polynomial, we get $p(A)\varphi(A) = (p\varphi)(A)$.

The idea is to let $\chi \in C_0(\mathbb{R})$ with $0 \le \chi \le 1$ be such that $\chi = 1$ on $\operatorname{supp}(\varphi) \cup \operatorname{supp}(\psi)$. Pick a sequence of polynomials $p_j$ such that $\overline{p}_j \chi \to \psi$ uniformly. Then $(p_j \chi)(A) \to \psi(A)$ in $\mathcal{L}(H, H)$. We will give the details next time. $\square$

# 22 Multiplicativity and the Functional Calculus for the Laplacian

## 22.1 Multiplicativity of the functional calculus

Last time, we were proving the multiplicativity of our functional calculus:

**Proposition 22.1.** *Let $A$ be self-adjoint, and let $\varphi, \psi \in C_0(\mathbb{R})$. Then $\varphi(A)\psi(A) = (\varphi\psi)(A)$.*

*Proof.* Last time, we showed that $\operatorname{Im}\varphi(A) \subseteq D(A^j)$ for all $j$ and that for any polynomial $p$, $p(A)\varphi(A) = (p\varphi)(A)$.

Let $\chi \in C_0(\mathbb{R})$ with $0 \leq \chi \leq 1$ and $\chi = 1$ on $\operatorname{supp}(\varphi) \cup \operatorname{supp}(\psi)$. For $u, v \in H$, write

$$\langle \varphi(A)u, (p\chi)(A)v\rangle = \langle \varphi(A)u, p(A)\chi(A)v\rangle$$

Since $\varphi \in D(A^j)$ for all $j$,

$$\begin{aligned} &= \langle \overline{p}(A)\varphi(A)u, \chi(A)v\rangle \\ &= \langle (\overline{p}\varphi)(A)u\chi(A)v\rangle\,. \end{aligned}$$

Take a sequence $p_j$ of polynomials such that $\chi\overline{p}_j \to \chi\psi = \psi$ uniformly on $\mathbb{R}$. Recall that we had for all $f \in C_B$, $\|f(A)\|_{\mathcal{L}(H,H)} \leq 2\|f\|_{L^\infty}$. Thus, $(\chi p_j)(A) \xrightarrow{\mathcal{L}(H,H)} \psi(A) = \psi(A)^*$. Also, $\overline{p}_j\varphi = \overline{p}_j\chi\varphi \to \psi\varphi$ uniformly, so $\overline{p}_j\varphi(A) \xrightarrow{\mathcal{L}(H,H)} (\psi\varphi)(A)$. We get

$$\langle \psi(A)\varphi(A)u, v\rangle = \langle \varphi(A)u, \psi(A)^*v\rangle = \langle (\psi\varphi)(A)u, \chi(A)v\rangle\,.$$

Now let $\chi \uparrow 1$ pointwise; we claim that $\chi(A) \to 1$ weakly. So we get $\langle \psi(A)\varphi(A)u, v\rangle = \langle (\psi\varphi)(A)u, v\rangle$ for all $u, v$.

To prove that $\chi(A) \to 1$ weakly, note that if $\varphi_j \in C_B$, $\varphi \in C_B$, and $\varphi_j \to \varphi$ pointwise boundedly ($\exists C$ such that $|\varphi_j(x)| \leq C$ for all $j, x$), then

$$\langle \varphi_j(A)u, v\rangle = \int \varphi_j(\lambda)\, d\mu_{u,v} \xrightarrow{j\to\infty} \int \varphi(\lambda)\, d\mu_{u,v}(\lambda) = \langle \varphi(A)u, v\rangle$$

by polarization and dominated convergence. □

## 22.2 Spectrum and functional calculus for the Laplacian

Let $A = -\Delta$ on $L^2(\mathbb{R}^n)$ be self-adjoint with $D(A) = H^2(\mathbb{R}^n)$. Given $\varphi \in C_0(\mathbb{R})$, we compute $\varphi(A)$.

**Proposition 22.2.** $\operatorname{Spec}(A) = [0, \infty)$.

*Proof.* First, $\operatorname{Spec}(A) \subseteq [0, \infty)$, as $A \geq 0$: for $u \in D(A)$,

$$\langle Au, u \rangle = \int |\nabla u|^2 \geq 0.$$

To get equality, it suffices to show that $(0, \infty) \subseteq \operatorname{Spec}(A)$ (as the spectrum is closed). For contradiction, let $\lambda > 0$ be such that $A - \lambda : D(A) \to L^2$ is bijective. Then there exists a constant $C > 0$ such that $\|u\|_{L^2} \leq C\|(A - \lambda)u\|_{L^2}$ for any $u \in D(A)$.

**Remark 22.1.** $A$ has no eigenvalues:[7] If $u \in L^2$ and $(-\Delta - \lambda)u = 0$, then taking the Fourier transform, we get

$$(|\xi|^2 - \lambda)\widehat{u}(\xi) = 0,$$

so $\widehat{u} = 0 \implies u = 0$.

Instead, we want to find **generalized eigenfunctions** $u \in L^\infty$ such that $(-\Delta - \lambda)u = 0$. We can take $u(x) = e^{ix\cdot\xi}$ for $\xi \in \mathbb{R}^n$ (where $|\xi|^2 = \lambda$). Consider the **quasimodes**[8] $u_j(x) = j^{-n/2}\chi(x/j)e^{ix\xi}$, where $\chi \in C_0^\infty(\mathbb{R}^n)$ is 1 near 0 with $\|\chi\|_{L^2} = 1$. Then $\|u_j\|_{L^2} = 1$, and

$$\|(A - \lambda)(j^{-n/2}\chi(x/j)e^{ix\cdot\xi})\|_{L^2} = O(1/j).$$

So the lower bound inequality for $A - \lambda$ cannot hold.

To determine $\varphi(A)$ for $\varphi \in C_0(\mathbb{R})$, notice that $\varphi(A) = 0$ if $\operatorname{supp}(\varphi) \subseteq (-\infty, 0)$. Compute the resolvent first: If $\operatorname{Im} z \neq 0$,

$$R(z)u = v, \quad u, v \in L^2 \iff (-\Delta - z)v = u$$

By Fourier transform, we get

$$R(z)u = \mathcal{F}^{-1}\left(\frac{\widehat{u}(\xi)}{|\xi|^2 - z}\right).$$

We get

$$\langle \varphi(A)u, u \rangle = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int \varphi(\lambda) \left\langle (R(\lambda + i\varepsilon) - R(\lambda - i\varepsilon))u, u \right\rangle \, d\lambda$$

By Parseval,

$$= \lim_{\varepsilon \to 0^+} \frac{1}{(2\pi)^n} \lim_{\varepsilon \to 0^+} \iint \varphi(\lambda)|\widehat{u}(\xi)|^2 \left(\frac{1}{|\xi|^2 - \lambda - i\varepsilon} - \frac{1}{|\xi|^2 - \lambda + i\varepsilon}\right) d\lambda \, d\xi$$

Integrate first in $\lambda$ and send $\varepsilon \to 0$ using dominated convergence.

$$= \frac{1}{(2\pi)^n} \int \varphi(|\xi|^2)|\widehat{u}(\xi)|^2 \, d\xi.$$

So we get

$$\varphi(A)u = \mathcal{F}^{-1}(\varphi(|\xi|^2)\widehat{u}).$$

□

[7] This says that the spectral measures have no pure point components.

[8] This terminology is common in mathematical physics literature.

## 22.3 Correcting the norm bound in the functional calculus

**Proposition 22.3.** *Let $A$ be self-adjoint, and let $\varphi \in C_0(\mathbb{R})$.*

$$\|\varphi(A)\|_{\mathcal{L}(H,H)} \leq \|\varphi\|_{L^\infty}.$$

Previously, we had a factor of 2 in the bound.

*Proof.* We have

$$\begin{aligned}
\|\varphi(A)u\|^2 &= \langle \varphi(A)u, \varphi(A)u \rangle \\
&= \langle \overline{\varphi}(A)\varphi(A)u, u \rangle \\
&= \left\langle |\varphi|^2(A)u, u \right\rangle \\
&= \int |\varphi|^2(\lambda)\, d\mu_u(\lambda) \\
&\leq \|\varphi\|_{L^\infty}^2 \|u\|^2.
\end{aligned}$$

□

We also get the following result.

**Corollary 22.1.**

$$\varphi(A)\|_{\mathcal{L}(H,H)} \leq \|\varphi\|_{L^\infty(\mathrm{Spec}(A))}.$$

Next, we will extend this multiplicativity property to more continuous functions.

# 23 Functional Calculus for Bounded Continuous Functions and Bounded Baire Functions

## 23.1 Approximation in the functional calculus

Let $\varphi, \psi \in C_0(\mathbb{R})$, and let $A$ be self-adjoint. Lsat time, we showed that $\varphi(A)\psi(A) = (\varphi\psi)(A)$ and that $\|\varphi(A)\|_{\mathcal{L}(H,H)} \leq \|\varphi\|_{L^\infty(\mathrm{Spec}(A))}$.

These properties extend to $\varphi, \psi \in C_B(\mathbb{R})$ by approximation (pick $\varphi \in C_0$ with $\varphi_j \to \varphi$ pointwise and boundedly to get $\varphi_j(A) \to \varphi(A)$ weakly). Notice also that if $\varphi_k, \varphi \in C_B$ with $\varphi_j \to \varphi$ pointwise boundedly, then $\varphi_j(A) \to \varphi(A)$ strongly: for all $u \in H$,

$$\begin{aligned}
\|\varphi_j(A)u - \varphi(A)u\|^2 &= \langle (\varphi_j - \varphi)(A)u, (\varphi_j - \varphi)(A)u \rangle \\
&= \int |\varphi_j(\lambda) - \varphi(\lambda)|^2 \, d\mu_n(\lambda) \\
&\to 0
\end{aligned}$$

by dominated convergence. (And if we have uniform approximation, we get operator norm convergence.)

## 23.2 Domain of $A$ in terms of spectral measures

Recall that if $u \in D(A)$, then $\int \lambda^2 \, d\mu(\lambda) < \infty$ and $\int \lambda^2 \, d\mu(\lambda) = \|Au\|^2$. Assume, conversely, that $\int \lambda^2 \, d\mu_u(\lambda) < \infty$ for $u \in H$. Let $\varphi \in C_0$ with $0 \leq \varphi \leq 1$, and write

$$A\varphi(A)u = \varphi_1(A)u,$$

where $\varphi_1(\lambda) = \lambda\varphi(\lambda)$. We get

$$\begin{aligned}
\|A\varphi(A)u\|^2 &= \langle \varphi_1(A)u, \varphi_1(A)u \rangle \\
&= \int \lambda^2 \varphi(\lambda)^2 \, d\mu_u(\lambda).
\end{aligned}$$

Let $\varphi_j \in C_0$ with $0 \leq \varphi_j \leq 1$ and $\varphi_j \uparrow 1$. Then $\varphi_j(A)u \to u$ in $H$. Also,

$$\|A(\varphi_j(A) - \varphi_k(A))u\|^2 = \int \lambda^2 (\varphi_j^2(\lambda) - \varphi_k^2(\lambda)) \, d\mu_u(\lambda) \xrightarrow{j,k\to\infty} 0$$

by dominated convergence, so $A\varphi_j(A)u \to v$ in $H$. $A$ is closed (as it is self-adjoint), so $u \in D(A)$, and $Au = \lim_{j\to\infty} A\varphi_j(A)u$.

**Remark 23.1.**

$$\|d\mu_{u,v}\| = \sup_{|\varphi| \leq 1} \underbrace{\left| \int \varphi \, d\mu_{u,v} \right|}_{\langle \varphi(A)u, v \rangle} \leq \|u\| \cdot \|v\|.$$

## 23.3 Summary of properties of the functional calculus

Let's summarize our results:

**Proposition 23.1** (continuous bounded functional calculus)**.** *Let $A$ be self-adjoint. The map $\Phi : C_B(\mathbb{R}) \to \mathcal{L}(H, H)$ sending $\varphi \mapsto \varphi(A)$ has the following properties:*

1. *$\Phi$ is an algebra homomorphism.*
2. *$\varphi(A)^* = \overline{\varphi}(A)$.*
3. *$\|\varphi(A)\| \leq \|\varphi\|_{L^\infty(\mathrm{Spec}(A))}$*
4. *For all $u \in H$, the map $d\mu_u : \varphi \mapsto \langle \varphi(A)u, u\rangle$ is a positive measure.*
5. *$1(A) = 1 \in \mathcal{L}(H, H)$.*[9]
6. *If $\varphi_j \to \varphi$ pointwise boundedly, then $\varphi_j(A) \to \varphi(A)$ strongly.*
7. *We have $D(A) = \{u \in H : \int \lambda^2 \, d\mu(\lambda) < \infty\}$, where $Au = \lim_{j\to\infty} A\varphi_j(A)u$ if $\varphi_j \uparrow 1$ and $\varphi_j \in C_0$.*

## 23.4 Extension of the functional calculus to bounded Baire functions

Next, we extend the functional calculus to the algebra of bounded Baire functions.

**Definition 23.1.** Let $K$ be a class of functions $f : \mathbb{R} \to \mathbb{C}$. We say that $K$ is **closed under pointwise limits** if for any sequence $f_j \in K$ such that $f = \lim_j f_j$ exists, we have $f \in K$. We let the **Baire functions** $\mathrm{Ba}(\mathbb{R})$ be the smallest class of functions $\mathbb{R} \to \mathbb{C}$ containing $C(\mathbb{R})$ which is closed under pointwise limits.

**Remark 23.2.** $\mathrm{Ba}(\mathbb{R})$ is an algebra under pointwise multiplication: Let $f \in C(\mathbb{R})$, and let $K = \{g : fg \in \mathrm{Ba}(\mathbb{R})\}$. $K \supseteq C(\mathbb{R})$ and is closed under pointwise limits, so $K \supseteq \mathrm{Ba}(\mathbb{R})$. Similarly, we extend to $f \in \mathrm{Ba}(\mathbb{R})$.

**Remark 23.3.** If $\mu$ is a positive (Radon) measure and $f \in \mathrm{Ba}(\mathbb{R})$, then $f$ is $\mu$-measurable.

Let $\mathrm{Ba}_b(\mathbb{R})$ be the Banach algebra of **bounded Baire functions** ($\mathrm{Ba}_b(\mathbb{R}) \subseteq L^1(\mu_u)$ for all $u \in H$).

**Proposition 23.2.** *Let $\varphi \in \mathrm{Ba}_b(\mathbb{R})$. There exists a unique, bounded linear map $\varphi(A) \in \mathcal{L}(H, H)$ such that*

$$\langle \varphi(A)u, v\rangle = \int \varphi(\lambda) \, d\mu_{u,v}(\lambda).$$

[9] This actually follows from the fact that $\Phi$ is an algebra homomorphism.

*Proof.* We have to check that $(u, v) \mapsto \int \varphi(\lambda)\, d\mu_{u,v}$ is sesquilinear; the Riesz-representation theorem will provide $\varphi(A)$. We may assume that $\varphi$ is real, so $|\varphi| \leq M$. Let us check that

$$\int \varphi(\lambda)\, d\mu_{\lambda_1 u_1 + \lambda_2 u_2, v} = \lambda_1 \int \varphi(\lambda)\, d\mu_{u_1,v}(\lambda) + \lambda_2 \int \varphi(\lambda)\, d\mu_{u_2,v}(\lambda).$$

Let $K = \{\varphi \in \mathrm{Ba}_b(\mathbb{R}; \mathbb{R}) : |\varphi| \leq M, \text{ this condition holds}\}$. Then $K$ contains continuous functions, and $K$ is closed under pointwise limits (by dominated convergence). We need one more claim:

Claim: The class $\{\varphi\, \mathrm{Ba}_b(\mathbb{R}; \mathbb{R}) : |\varphi| \leq M\}$ is the smallest class of functions $\mathbb{R} \to [-M, M]$ containing continuous functions $\mathbb{R} \to [-M, M]$ which is closed under pointwise limits. Check that this holds. We get that $K = \{\varphi \in \mathrm{Ba}_b(\mathbb{R}; \mathbb{R}) : |\varphi| \leq M\}$, and the proposition follows. $\square$

# 24 The Spectral Theorem for Unbounded, Self-Adjoint Operators

## 24.1 Multiplicative properties of the functional calculus for bounded Baire functions

Last time, we set about extending our functional calculus to the class of bounded **Baire functions** $\mathrm{Ba}(\mathbb{R})$, the smallest class of functions $\mathbb{R} \to \mathbb{C}$ containing $C(\mathbb{R})$ which is closed under pointwise limits. We showed that for any $\varphi \in \mathrm{Ba}_b(\mathbb{R})$, there is a unique map $\varphi(A) \in \mathcal{L}(H, H)$ such that $\langle \varphi(A)u, v\rangle = \int \varphi(\lambda)\, d\mu_{u,v}(\lambda)$.

As in the case for continuous functions, we get

$$\|\varphi(A)\| \leq \|\varphi\|_{L^\infty(\mathrm{Spec}(A))}.$$

If $\varphi \in \mathrm{Ba}_b(\mathbb{R})$ is real, then $\langle \varphi(A)u, u\rangle \in \mathbb{R}$, so $\varphi(A)$ is self-adjoint. In general, $\varphi(A)^* = \overline{\varphi}(A)$.

Next, we have the multiplicative property:

**Proposition 24.1.** *Let $A$ be self-adjoint, and let $\varphi, \psi \in \mathrm{Ba}_b(\mathbb{R})$. Then*

$$(\varphi\psi)(A) = \varphi(A)\psi(A).$$

*Proof.* We may assume that $\varphi, \psi$ are real. Fix $\varphi \in C_B$ and consider $K_M = \{\psi \in \mathrm{Ba}(\mathbb{R}; \mathbb{R}) : |\psi| \leq M, \text{mult. prop. holds}\}$. Then $K_M$ contains the continuous functions, and $K_M$ is closed under pointwise convergence: if $\psi_j \in K_M$ with $\psi_j \to \psi$ pointwise, then $\psi_j(A) \to \psi(A) \in K_M$ weakly, so $(\varphi\psi_j)(A) \to (\varphi\psi)$ weakly. It follows that $K_M = \{\psi \in \mathrm{Ba}(\mathbb{R}; \mathbb{R}) : |\psi| \leq M\}$. Next, keeping $\psi \in \mathrm{Ba}_b(\mathbb{R}; \mathbb{R})$ fixed, we extend the multiplicative property to all $\varphi \in \mathrm{Ba}_b$. $\square$

It follows as in the continuous case that if $\varphi_j \in \mathrm{Ba}_b(\mathbb{R})$ with $\varphi_j \to \varphi$ pointwise boundedly, then $\varphi_j(A) \to \varphi(A)$ strongly (for all $u \in H$, $\|\varphi_j(A)u - \varphi(A)u\| \to 0$).

**Remark 24.1.** Assume that $\varphi \in \mathrm{Ba}_b$ is such that $\psi(\lambda) = \lambda\varphi(\lambda) \in \mathrm{Ba}_b$. Let

$$r(\lambda) = \frac{1}{z - \lambda},$$

$$\operatorname{Im} z \neq 0, \qquad \varphi_z(\lambda) = r(\lambda)^{-1}\varphi(\lambda) = \psi(\lambda) - z\varphi(\lambda) \in \mathrm{Ba}_b\,.$$

We can write

$$\varphi(\lambda) = r(\lambda)\varphi_z(\lambda),$$

so

$$\varphi(A) = r(A)\varphi_z(A).$$

Now $r(A) = R(z)$ ($\lambda r(\lambda) \in C_B$, so $Ar(A) = \frac{\lambda}{\lambda - z}(A) = 1 + zr(A)$). We get that $\operatorname{Im}\varphi(A) \subseteq D(A)$ and

$$A\varphi(A) = \underbrace{AR(z)}_{!+zR(z)}\varphi_z(A) = \psi(A).$$

So we get that the multiplicative property holds when one of the functions is $\lambda$ (which is unbounded), as long as the result is still bounded.

## 24.2 The spectral theorem for unbounded, self-adjoint operators

**Theorem 24.1** (spectral theorem)**.** *Given a self-adjoint operator $A$, there is a unique algebra homomorphism $\Phi : \mathrm{Ba}_b(\mathbb{R}) \to \mathcal{L}(H, H)$ sending $\varphi \mapsto \varphi(A)$ such that:*

1. $\overline{\varphi}(A) = \varphi(A)^*$
2. $\|\varphi(A)\| \leq \|\varphi\|_{L^\infty(\mathrm{Spec}(A))}$
3. *If $\varphi_j \in \mathrm{Ba}_b$ with $\varphi_j \to \varphi$ pointwise boundedly, then $\varphi_j(A) \to \varphi(A)$ strongly.*
4. *If $\varphi, \varphi_1(\lambda)$ $(= \lambda\varphi(\lambda)) \in \mathrm{Ba}_b$, then $\varphi_1(A) = A\varphi(A)$.*
5. *$u \in D(A)$ if and only if there is a uniform upper bound for $\|\varphi_1(A)u\|$ when $\varphi_1(\lambda) = \lambda\varphi(\lambda)$, $0 \leq \varphi \leq 1$, and $\varphi \in \mathrm{Ba}_b$ with compact support. In this case,*

$$Au = \lim_{j\to\infty} \varphi_{1,j}(A)u,$$

*where $\varphi_j \uparrow 1$.*

*Proof.* Only uniqueness remains to be checked. Let $r_z(\lambda) = \frac{1}{\lambda - z}$, where $\operatorname{Im} z \neq 0$. Then $\lambda r_z(\lambda) = 1 + zr_z(\lambda)$, so by property 4, $A\Phi(r_z) = 1 + z\Phi(r_z)$. So we conclude that $\Phi(r_z) = R(z)$, the resolvent of $A$. Given $u \in H$, let $\mu$ be the measure on $\mathbb{R}$ sending $\varphi \mapsto \langle \Phi(\varphi)u, u\rangle$. Then

$$\langle \Phi(r_z)u, u\rangle = \langle R(z)u, u\rangle = \int \frac{1}{\lambda - z}\, d\mu(\lambda).$$

By the proof of Nevanlinna's theorem,

$$d\mu = \lim_{\varepsilon\to 0^+} \frac{1}{\pi} \operatorname{Im}\langle R(\lambda + i\varepsilon)u, u\rangle\; d\lambda,$$

which shows the uniqueness. $\square$

### 24.3 Extending the functional calculus to unbounded Baire functions

Next, we will define $\varphi(A)$ as an unbounded operator when $\varphi \in \mathrm{Ba}(\mathbb{R})$. Recall that if $\varphi \in \mathrm{Ba}_b$, then

$$\|\varphi(A)u\|^2 = \int |\varphi(\lambda)|^2 \, d\mu_u(\lambda).$$

Define

$$\begin{aligned} D(\varphi(A)) &= \{u \in H : \int |\varphi(\lambda)|^2 \, d\mu_i(\lambda) < \infty\} \\ &= \left\{ u \in H : \sup_{|\psi| \leq |\varphi|, \psi \in \mathrm{Ba}_b} \|\psi(A)u\| < \infty \right\}. \end{aligned}$$

This is a linear subspace of $H$.

To define $\varphi(A)$, let $\varphi_j \in \mathrm{Ba}_b$ be such that $\varphi_j \to \varphi$ with $|\varphi_j| \leq |\varphi|$. Then, for any $u \in D(\varphi(A))$, $\lim_{j \to \infty} \varphi_j(A)u$ exists ($\|\varphi_j(A)u - \varphi_k(A)u\|^2 = \int |\varphi_j - \varphi_k|^2 \, d\mu_i \xrightarrow{j,k \to \infty} 0$ by dominated convergence). So we let

$$\varphi(A)u = \lim \varphi_k(A)u,$$

which is independent of the chosen sequence.

**Proposition 24.2.** *Let $A$ be self-adjoint, and let $\varphi \in \mathrm{Ba}(\mathbb{R})$. Then $\varphi(A)$ is densely defined, and $\varphi(A)^* = \overline{\varphi}(A)$ (so $D(\varphi(A)^*) = D(\overline{\varphi})$). In particular, $\varphi(A)$ is closed.*

# 25 Projection-Valued Measures

## 25.1 Unbounded functional calculus

Let $A$ be self-adjoint, and let $\varphi \in \mathrm{Ba}(\mathbb{R})$. We want to define $\varphi(A)$ by

$$D(\varphi(A)) = \left\{ u \in H : \int |\varphi(\lambda)|^2 \, d\mu_u(\lambda) < \infty \right\},$$

$$\varphi(A)u = \lim_{j \to \infty} \varphi_j(A)u, \qquad \varphi_j \in \mathrm{Ba}_b(\mathbb{R}), \varphi_j \to \varphi, |\varphi_j| \leq |\varphi| < \infty.$$

**Proposition 25.1.** *Let $A$ be self-adjoint, and let $\varphi \in \mathrm{Ba}(\mathbb{R})$. Then $\varphi(A)$ is densely defined, and $\varphi(A)^* = \overline{\varphi}(A)$ (so $D(\varphi(A)^*) = D(\overline{\varphi})$). In particular, $\varphi(A)$ is closed.*

*Proof.* Let's check the first claim. If $u \in H$, let $\chi_n = \mathbb{1}_{\{|\varphi| \leq n\}}$. Then $\chi_n \in \mathrm{Ba}_b$, and $\chi_n \to 1$. So $\chi_n(A)u \to u$. Notice that if $u \in H$ and $\varphi \in \mathrm{Ba}_b(\mathbb{R})$, then $d\mu_{\varphi(A)u} = |\varphi|^2 \, d\mu_u$: for any $f \in C_0$,

$$\int f \, d\mu_{\varphi(A)u} = \langle f(A)\varphi(A)u, \varphi(A)u \rangle = \big\langle (f|\varphi|^2)(A)u, u \big\rangle = \int f|\varphi|^2 \, d\mu_u.$$

In particular, $\mathrm{supp}(d\mu_{\chi_n(A)u}) \subseteq \{|\varphi| \leq n\}$, so

$$\int |\varphi|^2 \, d\mu_{\chi_n(A)u} < \infty,$$

which implies that $\chi_n(A)u \in D(\varphi(A))$. Now use this argument with $\chi_n \varphi$ to get the claim. $\square$

## 25.2 Projection-valued measures

Let $M \subseteq \mathbb{R}$ be a Baire set ($\mathbb{1}_M \in \mathrm{Ba}$) and set $E(M) := \mathbb{1}_M(A) \in \mathcal{L}(H, H)$. Then

- $E(M)$ is an orthogonal projection on $H$.
- $E(\mathbb{R}) = 1$.
- $E(M_1)E(M_2) = E(M_1 \cap M_2)$.
- $E(\varnothing) = 0$.
- If $M = \bigcup_{j=1}^\infty M_j$ with $M_j \cap M_k = \varnothing$, then $E(M)$ is the strong limit of $\sum_{j=1}^N E(M_j)$ as $N \to \infty$.

The map $M \mapsto E(M)$ is an **orthogonal projection-valued measure**.

Recall next that if $\mu$ is a positive bounded measure on $\mathbb{R}$, then there exists a unique left continuous increasing function $\varphi$ such that $\varphi(\lambda) \to 0$ as $\lambda \to -\infty$ and $\int f\,d\mu = \int f\,d\varphi$ for all $f \in C_0$: We have $\varphi(\lambda) = \mu(\mathbb{1}_{(-\infty,\lambda)})$.

In particular,

$$\mu_u(\mathbb{1}_{(-\infty,\lambda)}) = \left\langle \mathbb{1}_{(-\infty,\lambda)}(A)u, u \right\rangle = \langle E_\lambda u, u\rangle\,, \qquad \text{where } E_\lambda = E((-\infty,\lambda)) = \mathbb{1}_{(-\infty,\lambda)}(A).$$

We have:

- $E_\lambda u \to 0$ as $\lambda \to -\infty$,
- $E_\lambda u \to u$ as $\lambda \to \infty$,
- $E_{\lambda-\varepsilon} u \to E_\lambda u$ as $\varepsilon \to 0^+$.

If $\varphi \in \mathrm{Ba}_b$, we have

$$\langle \varphi(A)u, u\rangle = \int \varphi(\lambda)\, d\mu_u(\lambda) = \int \varphi(\lambda)\, d\langle E_\lambda u, u\rangle.$$

If $u \in D(A)$, then $Au = \lim_{j\to\infty} A\varphi_j(A)u$, where $0 \le \varphi_j \le 1$ with $\varphi_j \in \mathrm{Ba}$ and $\mathrm{supp}(\varphi_j)$ compact. Then

$$\langle Au, u\rangle = \lim_{j\to\infty} \int \lambda\varphi_j(\lambda)\, d\langle E_\lambda u, u\rangle = \int \lambda\, d\langle E_\lambda u, u\rangle.$$

Formally, we write

$$A = \int \lambda\, dE_\lambda.$$

## 25.3 Properties of projection-valued measures

There is nothing new here, but we are taking a different (and useful) point of view. Here is a proposition that expresses this point of view.

**Proposition 25.2.**

1. *Let $\lambda \in \mathbb{R}$. Then $\lambda \in \mathrm{Spec}(A)$ if and only if $E((\lambda-\varepsilon, \lambda+\varepsilon)) \neq 0$ for all $\varepsilon > 0$.*
2. *$E(\{\lambda\}) \neq 0$ if and only if $\lambda$ is an eigenvalue of $A$ and $E(\{\lambda\})$ is the orthogonal projection onto $\ker(A-\lambda)$.*

*Proof.*

1. ( $\Longleftarrow$ ): For any $\varepsilon > 0$, pick $u_\varepsilon \in H$ such that $\|u_\varepsilon\| = 1$ and $u_\varepsilon = E((\lambda-\varepsilon, \lambda+\varepsilon))u_\varepsilon$. Then $d\mu_{u_\varepsilon}(t) = \mathbb{1}_{(\lambda-\varepsilon,\lambda+\varepsilon)}\, d\mu_{u_\varepsilon}(t)$. So $u_\varepsilon \in D(A)$ and

$$\|(A-\lambda)u_\varepsilon\| = \int (t-\lambda)^2\, d\mu_{u_\varepsilon}(t) \le \varepsilon^2 \int d\mu_{u_\varepsilon}(t) = \varepsilon^2$$

It follows that $\lambda \in \mathrm{Spec}(A)$: for any $\varepsilon > 0$ there is a $u_\varepsilon \in H$ such that $\|u_\varepsilon\| = 1$ and $\|(A-\lambda)u_\varepsilon\| \le \varepsilon$, so $A - \lambda$ has no bounded inverse.

( $\Longrightarrow$ ): If $E((\lambda-\varepsilon, \lambda+\varepsilon)) := E(I_\varepsilon) = 0$ for some $\varepsilon > 0$, then for any $u \in H$,

$$u = E(\mathbb{R})u = E(\mathbb{R} \setminus I_\varepsilon)u$$

Then for any $u \in H$, $\mathrm{dist}(\lambda, \mathrm{supp}(\mu_u)) \ge \varepsilon$. Let $\varphi(t) = \frac{1}{t-\lambda}$, which is continuous and bounded on $\mathrm{supp}(\mu_u)$ for all $u$. Then define $\varphi(A) \in \mathcal{L}(H,H)$ by $\langle \varphi(A)u, u\rangle = \int \varphi(t)\, d\mu_u(t)$. We get $(A-\lambda)\varphi((A) = 1$, so $\lambda \notin \mathrm{Spec}(A)$.

2. ( $\Longrightarrow$ ): If $E(\{\lambda_0\}) \neq 0$, then let $u \neq 0$ be such that $u = E(\{\lambda_0\})u$. Then $\mathrm{supp}(\mu_u(\lambda)) \subseteq \{\lambda_0\}$, so $u \in D(A)$ and

$$\|(A-\lambda_0)u\|^2 = \int (\lambda-\lambda_0)^2\, d\mu_u(\lambda) = 0.$$

So $\lambda_0$ is an eigenvalue, and $\mathrm{im}\, E(\{\lambda_0\}) \subseteq \ker(A - \lambda - 0)$.

On the other hand, if $(A-\lambda_0)u = 0$ for some $0 \neq u \in D(A)$, then for $\mathrm{Im}\, z \neq 0$, $(A-z)u = (\lambda_0 - z)u$, so

$$R(z)u = \frac{1}{\lambda_0 - z}u, \qquad \langle R(z)u, u\rangle = \frac{\|u\|^2}{\lambda_0 - z}.$$

So by the uniqueness in our Nevanlinna representation, we get $\mu_u(\lambda) = \|u\|^2\delta(\lambda-\lambda_0)$. So for any $\varphi \in \mathrm{Ba}_b$, $\varphi(A)u = \varphi(\lambda_0)u$. So $E(\{\lambda_0\}) \neq 0$, and $E(\{\lambda_0\})u = u$. Thus, $\mathrm{im}\, E(\{\lambda_0\}) = \ker(A - \lambda_0)$. $\square$

# 26 Weyl's Criterion and Weyl's Theorem

## 26.1 Weyl's criterion

Last time, we had $\operatorname{Spec}(S) = \operatorname{Spec}_d(A) \sqcup \operatorname{Spec}_{\text{ess}}(A)$, where $\operatorname{Spec}_d(A)$ is the set of isolated eigenvalues of $A$ of finite multiplicity.

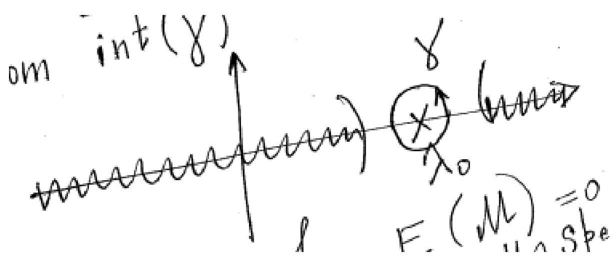
**Proposition 26.1** (Weyl's criterion)**.** *$\lambda \in \operatorname{Spec}_{\text{ess}}(A)$ if and only if there exists a sequence $u_n \in D(A)$ with $\|u_n\| = 1$, such that $u_n \to 0$ weakly and $\|(A - \lambda)u_n\| \to 0$.*

Such a sequence is called a **Weyl sequence**.

**Lemma 26.1.** *Let $\lambda$ be an isolated point of $\operatorname{Spec}(A)$. Then $\lambda_0$ is an eigenvalue, and*

$$E(\{\lambda_0\}) = \frac{1}{2\pi i} \int_\gamma (z - A)^{-1}\, dz,$$

*where $\gamma$ is a small circle centered at $\lambda_0$ with $\operatorname{Spec}(A) \setminus \{\lambda\}$ away from $\overline{\operatorname{int}(\gamma)}$.*




*Proof.* $E(M) = 0$ if $M \cap \operatorname{Spec}(A) = \varnothing$. For all small enough $\varepsilon > 0$, we have

$$0 \neq E((\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)) = E(\{\lambda_0\}),$$

so $\lambda_0$ is an eigenvalue. Compute

$$\begin{aligned}\frac{1}{2\pi i} \int_\gamma \langle (z - A)^{-1} u, u\rangle\, dz &= \frac{1}{2\pi i} \int_\gamma \int \frac{1}{z - \lambda}\, d\langle E_\lambda u, u\rangle\, dz \\ &= \int \left( \frac{1}{2\pi i} \int_\gamma \frac{1}{z - \lambda}\, dz \right) d\langle E_\lambda u, u\rangle\end{aligned}$$

The part in the parentheses equals 1 if $\lambda \in \operatorname{int}(\gamma)$ and 0 if $\lambda \notin \overline{\operatorname{int}(\gamma)}$.

$$\begin{aligned} &= \int \mathbb{1}_{\operatorname{int}(\gamma) \cap \mathbb{R}}(\lambda)\, d\langle E_\lambda u, u\rangle \\ &= \langle E(\{\lambda_0\}) u, u\rangle \\ &= \langle E(\operatorname{int}(\gamma)) u, u\rangle. \end{aligned}$$

□

Now we can prove Weyl's criterion.

*Proof.* ( $\Longleftarrow$ ): Assume that there is a sequence $u_n \in D(A)$ with $\|u_n\| = 1$, such that $u_n \to 0$ weakly and $\|(A-\lambda)u_n\| \to 0$. Then $\lambda \in \mathrm{Spec}(A)$. Assume that $\lambda \in \mathrm{Spec}_d(A)$. Then, using

$$(z-A)u_n = (\lambda - A)u_n) + (z-\lambda)u_n,$$

we have

$$(z-A)^{-1}u_n = \frac{1}{z-\lambda}u_n - (z-A)^{-1}\frac{1}{z-\lambda}(\lambda - A)u_n.$$

Integrate over $\gamma$ to get

$$E(\{\lambda\})u_n = u_n - \underbrace{\frac{1}{2\pi i}\int_\gamma (z-A)^{-1}(z-\lambda)^{-1}(\lambda-A)u_n\,dz}_{\to 0 \text{ in } \mathcal{H}}.$$

We get that $\|E(\{\lambda\})u_n - u_n\| \to 0$. $E(\{\lambda\})$ is finite rank, so it is compact. Therefore, $E(\{\lambda\})u_n \to 0$, which contradicts the fact that $\|u_n\| = 1$.

( $\Longrightarrow$ ): Let $\lambda \in \mathrm{Spec}_{\mathrm{ess}}(A)$. We claim that for all $\varepsilon > 0$, $\dim E((\lambda-\varepsilon, \lambda+\varepsilon))\mathcal{H} = \infty$. Indeed, assume that $\dim E(I)\mathcal{H} < \infty$ for some $I = (\lambda-\varepsilon, \lambda+\varepsilon)$. Then write

$$\mathcal{H} = E(I)\mathcal{H} \oplus E(I^c)\mathcal{H},$$

which is a closed, orthogonal direct sum. This composition reduces $A$, and $\mathrm{Spec}(A|_{E(I)\mathcal{H}}) \cap I \neq \varnothing$. Hence, $\mathrm{Spec}(A) \cap I = \mathrm{Spec}(A|_{E(I)\mathcal{H}})$, which is finite, of finite multiplicity. So $\lambda \in \mathrm{Spec}_d(A)$.

If $\dim E(\{\lambda\})\mathcal{H} = \infty$ ($\lambda$ is an eigenvalue of $\infty$ multiplicity), we let $u_n \in E(\{\lambda\})\mathcal{H}$ be an orthonormal sequence. In general, we can find a sequence $\varepsilon_n \downarrow 0$ such that

$$P_n = E((\lambda - \varepsilon_n, \lambda - \varepsilon_{n+1}) \cup (\lambda + \varepsilon_{n+1}, \lambda - \varepsilon_n)) \neq 0, \qquad P_n P_m = 0, \quad n \neq m.$$

It suffices to take $u_n \in P_n\mathcal{H}$ of norm 1. $\square$

## 26.2 Weyl's theorem

**Theorem 26.1** (Weyl)**.** *Let $A, B$ be self-adjoint, bounded from below, and such that for some $c \in \mathbb{R}$, $(A+c)^{-1} - (B+c)^{-1}$ is compact. Then* $\mathrm{Spec}_{\mathrm{ess}}(A) = \mathrm{Spec}_{\mathrm{ess}}(B)$.

*Proof.* We check that $\mathrm{Spec}_{\mathrm{ess}}(A) \subseteq \mathrm{Spec}_{\mathrm{ess}}(B)$. Let $\lambda \in \mathrm{Spec}_{\mathrm{ess}}(A)$, let $u_n$ be a Weyl sequence for $A$ at $\lambda$, and let $v_n = (B+c)^{-1}(A+c)u_n \in D(B)$. Write $(B+c)^{-1} = (A+c)^{-1} + K$, where $K$ is compact. Then

$$v_n = u_n + K(A+c)u_n$$

$$= u_n + \underbrace{K(A-\lambda)u_n}_{\to 0} + \underbrace{K\underbrace{(\lambda+c)u_n}_{\to 0 \text{ weakly}}}_{\to 0}.$$

So $\|v_n\| \geq 1/2$ for all large $n$. Hence, $w_n = \frac{v_n}{\|v_n\|} \in D(B)$, $w_n \to 0$ weakly, and $(B-\lambda)w_n \to 0$ since

$$\begin{aligned}(B-\lambda)v_n) &= (B+c)v_n - (-\lambda - c)v_n \\ &= (A+c)u_n - (\lambda+c)v_n \\ &= \underbrace{(A-\lambda)u_n}_{\to 0} - (\lambda+c)\underbrace{(v_n-u_n)}_{\to 0}.\end{aligned}$$

□

## 26.3 Applications

**Example 26.1.** Consider $P_0 = -\Delta$ on $\mathbb{R}^n$, and let $P = P_0 + q$ with $q \in C(\mathbb{R}^n;\mathbb{R})$ such that $q \to 0$ as $|x| \to \infty$. $P$ and $P_0$ are self-adjoint with $D(P) = D(P_0) = H^2(\mathbb{R}^n)$, and let us check that $(P+c)^{-1} - (P_0+c)^{-1}$ is compact:

Let us write $(P+x)u = (P_0+c)u + qu$ and replace $u \in H^2$ by $(P_0+c)^{-1}u$ for $u \in H^2(\mathbb{R}^n)$. Then $u \in L^2$ with

$$(P+c)(P_0+c)^{-1}u = u + q(P_0+c)^{-1}u.$$

Apply $(P+c)^{-1}$ to get

$$(P_0+c)^{-1}u = (P+c)^{-1}u + (P+c)^{-1}q(P_0+c)^{-1}u.$$

We get the resolvent identity

$$(P+c)^{-1} - (P_0+c)^{-1} = -(P+c)^{-1}q(P_0+c)^{-1}.$$

Let us check that $q(P_0+c)^{-1} : L^2 \to L^2$ is compact: Approximating $q$ by a sequence $q_j \in \overline{C_0}$ (uniformly), we may assume that $q \in C_0(\mathbb{R})$. If $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi q = q$, we get

$$q(P_0+c)^{-1} = q(\chi(P_0+c)^{-1}) : L^2 \to H^2 \cap \mathcal{E}(\operatorname{supp}(\chi)) \to L^2,$$

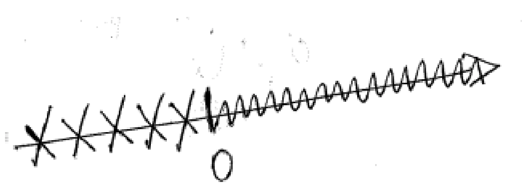
where the second map is compact. We get

$$\operatorname{Spec}_{\text{ess}}(P) = \operatorname{Spec}_{\text{ess}}(P_0) = [-\infty),$$

so

$$\operatorname{Spec}(P) = [0,\infty) \cup \{\text{negative eigenvalues}\},$$

where negative eigenvalues may only accumulate at 0.



If $q$ is rapidly decreasing at $\infty$, then $\operatorname{Spec}(P) \cap (-\infty)$ is finite and

$$\# \text{ of negative eigenvalues} \leq C_n \int |q|^{n/2}\, dx.$$

For $n \geq 3$, this is the Cwikel-Lieb-Rozenblum estimate (1972-1977). Observe that there are no negative eigenvalues if $q$ is small.

**Example 26.2.** Let $P = -\Delta + q$ with $q \in C(\mathbb{R}^n)$ and $q \geq 0$. Let $\eta = \liminf_{|x| \to \infty} q(x) \in [0, \infty]$. We claim that $\inf \operatorname{Spec}_{\text{ess}}(P) \geq \eta$ ($\operatorname{Spec}(P)$ is *discrete* in $(0, \eta)$). We may assume that $\eta < \infty$. Let

$$A = -\Delta + \max(q, \eta), \qquad B = q - \max(q, \eta) \in C(\mathbb{R}^n).$$

Then $B(x) \to 0$ as $|x| \to +\infty$. We get that $B(A + c)^{-1}$ is compact on $L^2$, since $(A + c)^{-1}(L^2) \subseteq H^1(\mathbb{R}^n)$. Thus,

$$\operatorname{Spec}_{\text{ess}}(\underbrace{-\Delta + q}_{=A+B}) = \operatorname{Spec}_{\text{ess}}(A) \subseteq \operatorname{Spec}(A) \subseteq [\eta, \infty).$$