Stat 155 Lecture 3 Notes

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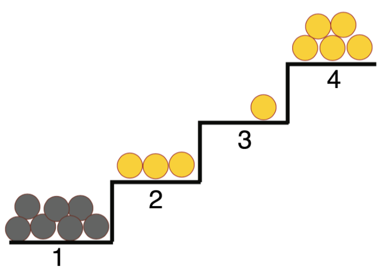
1 Staircase Nim and Partisan Games

1.1 Staircase Nim

Here is a game called Staircase Nim. Imagine a staircase with balls on the steps.[1] Every turn, a player takes some (positive) number of balls from a step and moves these balls down one step on the staircase. The player who moves the last ball to the bottom step wins. This game is progressively bounded.



Proposition 1.1. *Staircase Nim is equivalent to Nim, in the sense that we can define a mapping* $\phi : X \to X_{Nim}$*, such that* $P = \{x \in X : \phi(x) \in P_{Nim}\}$.

Proof. For a position $x = (x_1, x_2, \ldots, x_k)$ (number of chips on each step), define $\phi(x) = (x_2, x_4, \ldots, x_{2\lfloor k/2 \rfloor})$ (the number of chips on the even steps). Define the set

$$Z := \{x \in X : \phi(x) \in P_{\text{Nim}}\} = \left\{(x_1, \ldots, x_k) \in X : x_2 \oplus x_4 \oplus \cdots \oplus x_{2\lfloor k/2 \rfloor} = 0\right\}.$$

We want to show that $Z = P$. It is sufficient to show that

1. If $x \in Z$, $x' \in X \setminus Z$ for all x' such that $(x, x') \in M$.
2. If $x \in X \setminus Z$, $\exists x' \in Z$ such that $(x, x') \in M$.

[1]These figures for Staircase Nim are modified versions of figures from the book *Game Theory, Alive* by Anna Karlin and Yuval Peres.

If we move balls from an even to an odd step, say we move from state x to x'. This just decreases one of the components in the vector x, so it corresponds to a Nim move. So if $\phi(x)$ has 0 Nim-sum, $\phi(x')$ has nonzero Nim-sum. If we move balls from an even to an odd step, we increase the value of one of the piles in $\phi(x)$. This changes at least one place in the Nim sum, making $\phi(x')$ have nonzero Nim-sum. So every move in Z leads to a move in $X \setminus Z$.

If we start from x, where $\phi(x) \neq 0$, then there is some move in Nim that makes the Nim-sum 0. We can make this move in Staircase Nim by taking balls on an even step and moving them to an odd step. So for every $x \in X \setminus Z$, there is a move (x, x') such that $x' \in Z$. □

1.2 Partisan Games

1.2.1 Partisan subtraction game

Here is an partisan subtraction game. Start with 11 chips. Player 1 can remove wither 1 or 4 chips per turn. Player 2 can remove either 2 or 3 chips per turn. The game is played under normal play.

We can construct sets

$$N_i = \{\text{positions where Player } i\text{, playing next, can force a win}\},$$

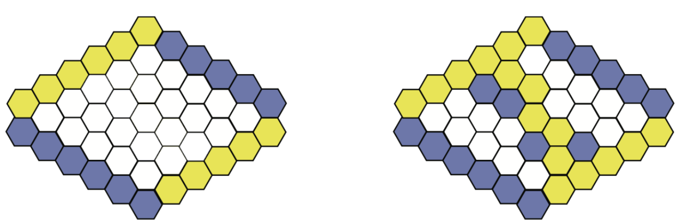
$$P_i = \{\text{positions where, if Player } i \text{ plays next, the previous player can force a win}\}.$$

In this game, $\{1, 2, 4, 5\} \subseteq N_1$, $\{2, 3, 5\} \subseteq N_2$, $\{0, 3\} \subseteq P_1$, and $\{0, 1, 4\} \subseteq P_2$.

Theorem 1.1. *Consider a progressively bounded partisan combinatorial game with no ties allowed. Then from any initial position, one of the players has a winning strategy.*

1.2.2 Hex

In the game of Hex, players alternate painting tiles on a board either yellow (Player 1) or blue (Player 2). The winner of the game is the first player who can construct a path from one side to the other.[2]



[2]These Hex diagrams are modified versions of diagrams from the book *Game Theory, Alive* by Anna Karlin and Yuval Peres.

This game is partisan because one player can only paint squares blue, and the other can only paint squares yellow. This game is progressively bounded because there are only finitely many tiles. Hex has no ties; this is nontrivial to prove, and we will not prove it here.

Theorem 1.2. *On a symmetric Hex board, the first player has a winning strategy.*

Proof. We use a strategy-stealing argument. Assume for the sake of contradiction that the second player has a winning strategy (i.e. a mapping S from the set of positions to the set of destinations of legal moves); we will construct a winning strategy for Player 1. The first player plays an arbitrary first move $m_{1,1}$. To play the n-th move, the first player calculates the position x_{n-1} of the board as if only moves $m_{2,1}, m_{2,2}, \ldots, m_{2,n-1}$ were played, and then plays $m_{1,n} = S_{\mathrm{rot}}(x_{n-1})$, where S_{rot} is the strategy S applied to the board rotated 90 degrees with colors switched. And if $m_{1,n}$ is not a legal move because that hexagon has already been played, choose something else; an extra hexagon can only help. So Player 1 also has a winning strategy. This is a contradiction, so Player 2 cannot have a winning strategy. So Player 1 has the winning strategy. □